Abstract of doctoral thesis

by Marta Leśniak

titled **"Normal generators of the mapping class group of a nonorientable surface"**

written under the supervision of prof. UG, dr hab. Błażej Szepietowski

The mapping class group of a compact, connected surface F is the group of isotopy classes of homeomorphisms of F equal to the identity on the boundary and orientation-preserving if F is orientable; we denote it by $\mathcal{M}(F)$. We call a surface of genus g with n border components $S_{g,n}$ if it is orientable and $N_{g,n}$ otherwise. If $n = 0$, we omit it, writing, for example N_g. The isotopy classes inherit the group operation from their representatives. The mapping class group plays an important role in several areas of mathematics, for example low-dimension topology, algebraic geometry, complex variable theory and geometric group theory.

The goal of this thesis is to investigate the problem of generating the mapping class group of a nonorientable surface by conjugates of one element, especially a torsion element, and to find examples of such generating sets. In Chapter 2 we show that one crosscap transposition normally generates the entire mapping class group, that is, the smallest normal subgroup of $\mathcal{M}(N_{g,n})$ containing the crosscap transposition is equal to the entire group. A crosscap transposition is a map on a nonorientable surface switching two crosscaps along a curve passing through them both.

One immediate consequence of this theorem is the fact that there exists a generating set of the mapping class group of a nonorientable surface consisting solely of crosscap transpositions; we show such a generating set for a nonorientable surface with at most one border component. This is a surprising result, because all the generating sets of $\mathcal{M}(N_{g,n})$ known previously consist of elements of two different topological types (two different conjugacy classes): Dehn twists and either y-homeomorphisms or crosscap transpositions. In the case of an orientable surface, by the results of Dehn, Lickorish and Mumford the mapping class group is normally generated by one Dehn twist.

The remainder of the theorems, described in Chapters 3 and 4, are concerned with torsion elements. By the results of Nielsen and Kerckhoff each torsion mapping class group $f \in \mathcal{M}(N_g)$ contains a representative

of the same order as f. In Chapter 3 we prove that for $g \geq 7$ the normal closure of any torsion element of order at least three contains the twist subgroup, or the subgroup of $\mathcal{M}(N_g)$ of index two generated by Dehn twists. As a corollary we give an example of a normal generator of $\mathcal{M}(N_g)$ for $g \geq 7$.

With that result we move on to involutions, or torsion elements of order two. To determine the conjugacy class of an involution on a nonorientable surface, we need up to six invariants, among them the fixed point set of the standard representative of the involution. We inspect each possible involution on N_g and find the necessary and sufficient conditions for its normal closure to contain the twist subgroup. Additionally, employing this result, we show an involution normally generating $\mathcal{M}(N_g)$.

In Chapter 4 we prove that $\mathcal{M}(N_g)$ is generated by three conjugate elements for $g \geq 12$. It is the first such a small torsion generating set of the mapping class group of a nonorientable surface. First we show that three conjugate elements of even order g generate the groups $\mathcal{M}(N_g)$ and $\mathcal{M}(N_{g+1})$ and their twist subgroups. An advantage of the above theorem is the small lower boundary on the genus, no complicated conditions and relatively easy proof. Next, for set even k, we define three conjugate elements of order k which normally generate $\mathcal{M}(N_g)$, supposing that g can be expressed as $a(k-1)+bk$ or $a(k-1)+bk+1$, where a is a non-negative integer and b an odd natural number. This condition is fulfilled for sufficiently large g.