Report on the doctoral dissertation of Adam Dzedzej, “Multi-state two-dimensional number-conserving cellular automata”

Henryk Fukś

August 8, 2019

The report is organized into several sections discussing various aspects of the dissertation. The last section, “concluding comments”, contains the final assessment of the dissertation.

Mastery of the subject matter

The candidate demonstrates very good understanding of the subject matter. Chapters 1 and 2 as well as sec. 3.1 provide concise introductory background on cellular automata (CA), general questions in the field, and conservation laws in CA. In particular, the major existing results on number-conserving cellular automata (NCCA) are summarized in ch. 3.1 and their meaning is clearly explained. Further discussion of more specialized prior research is given in the first sections of all remaining chapters. Appropriate references are provided. This shows the author's familiarity with the literature of the subject and his strong grasp of the theoretical foundations of the theory of CA. I also appreciate that the objectives of the dissertation are clearly spelled out in sec. 1.4.

Contribution of the dissertation to the advancement of knowledge

Contributions of the dissertation are described by the author in sec. 1.4 and furthermore summarized in the abstract and in chapter 8, thus I will not repeat all details here. I will, however, single out some contributions that I find especially valuable.

The first is the necessary and sufficient condition for number conservation in multi-state CA with the von Neumann neighbourhood in 2D, discussed in chapter 3. It is similar to the earlier result of Durand et al. (Theorem 3.1.2) for Moore's neigbourhood, but the author of the dissertation, unlike Durand (who provides only a sketch of the proof), gives fully complete proof based on three exquisite lemmas (3.2.1, 3.2.2 and 3.2.3). I found Lemma 3.2.3 especially striking and interesting, revealing the beautiful symmetry of NCCA. The aforementioned condition leads to the very elegant and simple characterization of special classes of NCCA, such as totalistic, outer-totalistic as well as symmetric and outer symmetric rules.

This will, I hope, inspire some further research, for example, on different neighbourhood shapes or on probabilistic 2D CA with conservation laws, as Theorem 3.2.4 and the method of its proof seem to be amenable to such generalizations.

The second major contribution of the dissertation are enumeration results. For three state NCCA the enumeration is a straightforward application of the results of chapter 3 on the necessary and sufficient condition for number conservation. The case of four states is more challenging - here the author uses the method of split functions developed by Wolnik et al. It is interesting to note that number-conserving rules are getting increasingly rare with the growing size of the alphabet (i.e., as a percentage of all rules). For two states the ratio of the number of NCCA rules to the number of all rules is $9/2^{2^5} \approx 2.1 \cdot 10^{-9}$. For three and four states this ratio is equal to, respectively, $1.5 \cdot 10^{-113}$ and $5.4 \cdot 10^{-610}$. One might conjecture that this ratio converges to zero as the number of states increases, as the convergence appears to be extremely fast. This observation is of course linked to Question 8.2 in chapter 8, asking for asymptotic bounds on the number of n-state NCCA.

In light of the above, it is thus not surprising that if one imposes additional constrains, the number of relevant NCCA can become nil. This is the case of three-state invertible NCCA rules (excluding trivial cases of identity and shifts). Only among rules with four states non-trivial invertible NCCA exist, and some examples are described at the end of chapter 6.

When instead of invertibility one requires rotational symmetry of the local function, there are no interesting cases among rules of up to 4 states. Only when the state count reaches 6, the number of rules becomes substantial (116), reaching 21280 rules for seven states. Enumeration results for rotation-symmetric NCCA, described in chapter 7, are the last but not least of all the enumeration results presented in the dissertation.

In summary, I have no doubt that the results presented in the dissertation significantly advance our knowledge of the distribution and of the frequency of occurrence of number conserving rules in the rule space of two-dimensional cellular automata.

Research methodology

The research methodology used in the dissertation follows the general practice used in the field. In general, cellular automata can exhibit an extensive diversity of dynamical behaviour, ranging from very simple (e.g., nilpotent rules) to immensely complex (e.g. elementary rule 110, or the "game of life"). It is very difficult, therefore, to say something meaningful about the behaviour of an *arbitrary* CA. One must instead consider somewhat restricted groups of CA, such as surjective rules, rules with additive invariants, symmetric rules, etc. A wealth of interesting mathematical results has been obtained for such restricted groups, and the same approach is used in the dissertation. The author considers rules which, in addition to being number-conserving, have fixed and small number of states and/or are also totalistic, outer-totalistic, invertible, rotation-symmetric. etc. He then tailors his methodology to each of these groups, in all cases very successfully, obtaining conditions which the local functions of the relevant groups must obey, and then enumerating them.

Organization, writing style and presentation of material

The dissertation is well organized and written in a very readable style. The material is presented in a coherent and orderly fashion. As a non-native speaker of English, I can say that the quality of English is very good (probably better than mine). I only have a couple of minor suggestions regarding the presentation, and these are included the next section.

Suggested improvements

1. One cannot help but notice the similarities of Theorems 3.1.3 and 3.2.4, but this does not seem to be mentioned in the dissertation. In particular, Theorem 3.1.3 could be written in another form, using b as the leading term instead of a (although I do not think this is explicitly stated in Durand's paper). One could then write the theorem for von Neumann neighbourhood by simply omitting "corner" sites, thus obtaining Theorem 3.2.4 (Lemma 3.2.1 would be needed to eliminate some terms). I would suggest to add a paragraph about this.

2. Sec. 5.3 refers to the "decomposition theorem", but no theorem seems to be explicitly named as such (theorem number should be added).

3. The idea of number-conservation for CA has been generalized to include, in addition to "classical" deterministic CA, also fuzzy CA (H. Betel and P. Flocchini. *On the relationship between boolean and fuzzy cellular automata.* Theoretical Computer Science, 412:703–713, 2011.) and probabilistic CA (H. Fukś. *Probabilistic cellular automata with conserved quantities.* Nonlinearity, 17:159–173, 2004). This could be mentioned somewhere, perhaps in chapter 8, as it is a possible avenue for generalization of results of the the dissertation.

4. By reading the first 20-something pages, the reader may get the (false) impression that only rules with a discrete set of states are discussed. Then suddenly $Q = [0, 1]$ appears on p. 29, in the context of totalistic rules, and one may think that this is a misprint. It would be helpful to remind the reader that in Theorem 3.3.1 and in what follows, the set Q could be any subset of real numbers containing 0. This may be a good idea especially because the rest of the dissertation (ch. 4-7) basically deals with the discrete states.

Concluding comments

The dissertation presents a significant body of original research, much of which has already been published or submitted for publication as articles in high-quality refereed research journals and conference proceedings. I should also add that Mr Dzedzej already made himself known in the world community of cellular automata researchers by actively participating and presenting his work in international conferences devoted to this subject. I can attest that his work and presentations have been very well received in the community.

In conclusion, Mr Dzedzej's work embodies substantial contribution to knowledge in the area of the theory of cellular automata, and without any doubt, it satisfies both statutory and customary requirements for a doctoral dissertation. I recommend that Mr. Adam Dzedzej is allowed to proceed to further stages of the doctoral process.

Moreover, given the high quality of the dissertation and scholarly achievements of the author as outlined above, I recommend the dissertation for distinction.

Finally, I would like to add that I greatly enjoyed reading this work, and I learned a few new things in the process. I wish Mr. Dzedzej all the best in his further scientific career.

Henryk Fukś
Professor
Department of Mathematics and Statistics
Brock University
St. Catharines, ON
Canada L2S 3A1
tel. (905) 688-5550, Ext. 3296
email:hfuks@brocku.ca