AUTOPRESENTATION

1. Basic information

1.1. **Names and surname: Błażej Jakub Szepietowski**

1.2. **Diplomas and degrees:**

- M.Sc. in Mathematics, Gdańsk University, Faculty of Mathematics and Physics, 2002;
- Ph.D. in Mathematics, Gdańsk University, Faculty of Mathematics, Physics and Informatics, 2006. Dissertation title: „Generators and relations in the mapping class group of a nonorientable surface". Supervisor: Prof. Grzegorz Gromadzki.

1.3. **Previous employment in scientific institutions:**

- 2002 - 2006, Assistant, Institute of Mathematics, Gdańsk University;
- Since 2006, Assistant Professor, Institute of Mathematics, Gdańsk University.

1.4. **The achievement according to art. 16.2 of the act of laws from 14 March 2003 on academic degrees:** a single-themed series of 5 publications entitled

„Mapping class group of a nonorientable surface".

Publications in the achievement:

[H1] B. Szepietowski, A presentation for the mapping class group of the closed non-orientable surface of genus 4, *Journal of Pure and Applied Algebra* 213 (2009), 2001–20016.

[H2] B. Szepietowski, Crosscap slides and the level 2 mapping class group of a nonorientable surface, *Geometriae Dedicata* 160 (2012), 169–183.

[H3] B. Szepietowski, A finite generating set for the level 2 mapping class group of a nonorientable surface, *Kodai Mathematical Journal* 36 (2013), 1–14.

[H4] B. Szepietowski, Low-dimensional linear representations of the mapping class group of a nonorientable surface, *Algebraic & Geometric Topology* 14 (2014), 2445–2474.

[H5] L. Paris, B. Szepietowski, A presentation for the mapping class group of a nonorientable surface, *Bulletin de la Société Mathématique de France* 143 (2015), 503–566.

2. Description of the scientific achievement

In addition to papers [H1-H5], some of my other papers [P1-P13] are also cited in this description. Their lists can be found in paragraphs 3.1 and 3.2. The list of papers of other authors is given at the end of this autopresentation.

2.1. **Introduction.**

The mapping class group of a compact connected nonorientable surface N, denoted by $\mathcal{M}(N)$, is the group of isotopy classes of homeomorphisms of N equal to the identity on the boundary ∂N if it is non-empty:

$$\mathcal{M}(N) = \mathrm{Homeo}(N, \partial N)/\mathrm{Homeo}_0(N, \partial N).$$

Here $\mathrm{Homeo}_0(N, \partial N)$ denotes the subgroup of $\mathrm{Homeo}(N, \partial N)$ consisting of the homeomorphisms isotopic to the identity, and by an isotopy we understand a homotopy $H\colon N \times [0,1] \to N$ such that $H(-,t) \in \mathrm{Homeo}(N, \partial N)$ for $t \in [0,1]$. The mapping class group of a compact connected orientable surface is defined analogously as the group of isotopy classes of orientation preserving homeomorphisms:

$$\mathcal{M}(S) = \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S).$$

When a finite set P of points is distinguished on the surface, then in the above definition we additionally assume that all homeomorphisms permute P, and we denote the mapping class group by $\mathcal{M}(N, P)$ or $\mathcal{M}(S, P)$.

A compact connected surface for which we neither assume that it is orientable nor nonorientable will be denoted by F, and its mapping class group by $\mathcal{M}(F)$ or $\mathcal{M}(F, P)$ in case of distinguishes points. We will also use the notation $N_{g,n}$, $S_{g,n}$, $F_{g,n}$ for a surface of genus g with n boundary components, dropping n if $n = 0$. Thus $N_{g,n}$ denotes a surface homeomorphic to the connected sum of g projective planes, from which the interiors of n pairwise disjoint discs have been removed.

Mapping class group plays a remarkably important role in low-dimensional topology (including the theory of 3- and 4-dimensional manifolds), the theory of functions of a complex variable, algebraic geometry and geometric group theory. It attracts great interest of many mathematicians and is an object of intense studies uninterruptedly for more than fifty years. Nevertheless, there are still many open problems related to this group.

The study of mapping class group was initiated in the 1920s independently by M. Dehn and J. Nielsen; but the truly dynamic development of this theory begun only in the 1960s and was propelled over the next decades by ground-breaking works of mathematicians such as W. B. R. Lickorish, J. S. Birman, W. P. Thurston, J. L. Harer, N. V. Ivanov, D. Johnson, B. Wajnryb. Theorems and methods developed by these authors are to this day basic tools in this field. Moreover, some of these methods, especially those coming from Thurston, have been successfully applied in the study of other, related groups, like the braid group and the group of automorphisms of a free group.

One of the reasons for the great importance of the group $\mathcal{M}(S_g)$ is its role in the construction of the moduli space of Riemann surfaces, where this group acts properly discontinuesly as the full isometry group of the Teichmüller space $\mathrm{Teich}(S_g)$, and the orbit space $\mathfrak{M}(S_g) = \mathrm{Teich}(S_g)/\mathcal{M}(S_g)$ of this action is the above-mentioned moduli space of compact Riemann surfaces of genus g ($g \geq 2$), a central object of the theory of functions of a complex variable and the theory of algebraic curves. By allowing antyholomorphic transition functions between charts one obtains the notion of a dianalitic structure of Klein surface on a nonorientable surface N_g. This concept was already considered by Klein himself. Its systematic description can be found in the modern monograph [1], and the methodology of their study was developed in [18]. The moduli space $\mathfrak{M}(N_g)$ of such structures is again the orbit space of the action of the mapping class group $\mathcal{M}(N_g)$ on the Teichmüller space $\mathrm{Teich}(N_g)$.

Every compact Klein surface is the orbit space $S/\langle\sigma\rangle$ for a unique pair (S, σ), where S is a Riemann surface, and $\sigma\colon S \to S$ its symmetry, that is an antyholomorphic involution. Under the well known functorial bijective correspondence between compact Riemann surfaces and smooth, irreducible, complex projective curves, symmetric surfaces correspond to curves having real equations. A pair (S, σ) is usually called a real algebraic curve [1].

Since $\mathrm{Teich}(F)$ is a manifold (homeomorphic to a ball in an euclidean space), $\mathfrak{M}(F)$ has the structure of an orbifold, whose singular points correspond to Riemann or Klein surfaces having nontrivial automorphisms. The group $\mathcal{M}(F)$ encodes most of the topological features of the space $\mathfrak{M}(F)$ and conversely, invariant such as the homology of $\mathcal{M}(F)$ are determined by the topology of $\mathfrak{M}(F)$. As examples of the above relationship let us mention the proofs of simple connectivity of the moduli spaces of Riemann and Klein surfaces [64], [P1], Harer's theorem [31] on stability of the (co)homology groups of $\mathcal{M}(S)$ and $\mathfrak{M}(S)$, or the Madsen-Weiss theorem [65] proving the Mumford's conjecture about the stable cohomology groups of $\mathfrak{M}(S)$. Analogous theorems for nonorientable surfaces were proved by N. Wahl [82].

The second, after the Teichmüller space, fundamental object on which the group $\mathcal{M}(F)$ acts is the curve complex $\mathcal{C}(F)$ defined by Harvey [35]. It is a simplicial complex, whose k-simplices are the isotopy classes of families of $k+1$ pairwise disjoint and pairwise nonisotopic simple closed curves on F. This complex pays a key role in the works of Harer [31, 32], Ivanov [43] and Wahl [82] concerning the (co)homology of $\mathcal{M}(F)$. After the proof of the hyperbolicity of $\mathcal{C}(S)$ by Masur and Minsky [66], the study of the mapping class group acquired a new dynamism. In our nonorientable case, the hyperbolicity of the curve complex $\mathcal{C}(N)$ was proved by Bestvina and Fujiwara [7] using the work of Bowditch [12], and also by Masur and Schleimer [67] by a different method. The involvement of the authors of this class in the studies indicates the rank of this subject. In the papers [H1, H5] we used the action of $\mathcal{M}(N)$ on the curve complex to find a finite presentation for this group.

The first papers devoted entirely to the mapping class group of a nonorientable surface were written already in the 1960s by Lickorish [61, 62], Chillingworth [19] and Birman-Chillingworth [9]. Then there was a thirty years long stagnation ended by the papers of Korkmaz [52, 53], and from that moment on the subject of the mapping class group of a nonorientable surface enjoys an increasing interest.

Every nonorientable surface N admits a covering of degree 2 by an orientable surface S. By the theorem of Birman and Chillingworth [9], the group $\mathcal{M}(N)$ is isomorphic to the subgroup of $\mathcal{M}(S)$ of infinite index consisting of the elements commuting with the covering involution. As a consequence of this relationship, some properties of $\mathcal{M}(S)$ automatically pass to $\mathcal{M}(N)$ - for example all kinds of residual properties. On the other hand, infiniteness of the index is a serious obstacle in problems such as, for examaple, finding a finite presentation. Thus, although the theorem of Birman-Chillingworth is very important, its usefulness is rather limited. Furthermore, many results about $\mathcal{M}(S)$ use the orientability in a fundamental way, so that their simple adaptation for the case of a nonorientable surface is impossible and new ideas are needed.

Many important theorems about $\mathcal{M}(S)$ have got their counterparts for a nonorientable surface proven, like the above-mentioned theorems of Harer, Madsen-Weiss and Masur-Minsky, or the no less famous theorem of Ivanov [46] about the automorphism group of $\mathcal{C}(S)$, which has been recently transplanted to nonorientable surfaces by Atalan and Korkmaz [3]. Until recently, one of the major exceptions to the above rule was Wajnryb's theorem [83, 86] providing a simple presentation for $\mathcal{M}(S)$ by generators and relations. The lack of such a presentation for the group $\mathcal{M}(F)$ was filled in the paper [H5], which I consider as my most important achievement.

I close this introduction with a short description of my main results obtained in the papers [H1-H5], in order of their importance in my opinion.

- The papers [H1, H5] are devoted to the problem of finding a finite presentation for the groups $\mathcal{M}(N_{g,n})$. In [H1] I found such a presentation for $(g,n) = (4,0)$, and in [H5], jointly with L. Paris, for $n \in \{0,1\}$ and all g such that $g+n > 3$. In the problem of obtaining finite presentations for $\mathcal{M}(N_{g,n})$ the most significant case is $n = 0$, because starting from a presentation of $\mathcal{M}(N_{g,0})$ one can inductively calculate presentations of $\mathcal{M}(N_{g,n})$ for all n by a method based on the Birman exact sequence, as in the paper [60] in the case of orientable surfaces.

- In the paper [H4] I described all nontrivial homomorphisms $\mathcal{M}(N_g) \to \mathrm{GL}(m,\mathbb{C})$ for $g \geq 5$ and $m \leq g-1$. In this way I extended, to the case of a nonorientable surface, the results recently obtained by J. Franks, M. Handel and M. Korkmaz, and completed the understanding of low-dimensional linear representations of mapping class groups of surfaces. The paper [H4] represents a significant contribution towards such understanding, because for nonorientable surfaces the situation is more complicated than for orientable ones. As an application, I proved that for $h < g$ and $g \geq 5$ any nontrivial homomorphism $\mathcal{M}(N_g) \to \mathcal{M}(N_h)$ has the image isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, where the latter case is possible only for $g \in \{5,6\}$.

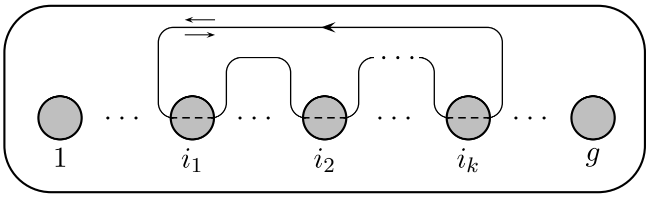

FIGURE 1. The curve γ_I for $I = \{i_1, i_2, \ldots, i_k\}$.

- The papers [H2, H3] are devoted to the level 2 mapping class group, denoted by $\Gamma_2(N_g)$ and defined as the subgroup of $\mathcal{M}(N_g)$ consisting of the isotopy classes of homeomorphisms inducing the identity on $H_1(N_g, \mathbb{Z}_2)$. In [H2] I proved that $\Gamma_2(N_g)$ is generated by so-called Y-homeomorphisms defined by Lickorish in 1963, and also that it is generated by involutions (elements of order 2). In [H3] I found a finite generating set for this group.

In the following I will describe the above results in more detail, on the background of works of other authors.

2.2. **Presentation by generators and relations.** [H1, H5]

McCool [70] gave the first algorithm for finding a finite presentation for $\mathcal{M}(S_{g,1})$ for any g. His approach is purely algebraic and no explicit presentation has been derived from this algorithm. In their ground-breaking paper [37] Hatcher and Thurston gave an algorithm for computing a finite presentation for $\mathcal{M}(S_{g,1})$ from its action on a certain simply connected 2-dimensional CW-complex. By this algorithm, Harer [30] obtained a finite, but very unwieldy, presentation for $\mathcal{M}(S_{g,1})$ for any g. This presentation was simplified by Wajnryb [83, 86], who also found a presentation for $\mathcal{M}(S_{g,0})$. Using Wajnryb's result, Matsumoto [68] obtained other presentations for $\mathcal{M}(S_{g,1})$ and $\mathcal{M}(S_{g,0})$, and Gervais [26] found a presentation for $\mathcal{M}(S_{g,n})$ for arbitrary $g \geq 1$ and n. Labruère and Paris [60] computed a finite presentation for $\mathcal{M}(S_{g,n}, P)$ for arbitrary $g \geq 1$, n and P. Benvenuti [6] and Hirose [38] independently recovered the Gervais presentation from the action of $\mathcal{M}(S_{g,n})$ on the Harvey's curve complex [35], instead of the Hatcher-Thurston complex.

Before the papers [H1, H5] finite presentations of $\mathcal{M}(N_{g,n})$ were known only for a few nonorientable surfaces of genus $g \leq 3$, including $\mathcal{M}(N_{2,0}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ [61] and $\mathcal{M}(N_{3,0}) \cong \mathrm{GL}(2, \mathbb{Z})$ [9, 27]. Using results of Lickorish [61, 62], Chillingworth [19] found a finite generating set for $\mathcal{M}(N_{g,0})$ for all $g \geq 3$. This result was extended to nonorientable surfaces with distinguished points [53] and boundary [77].

In order to formulate the main result of the papers [H1, H5] let us fix a model of a nonorientable surface. For $N_{g,1}$ (respectively $N_{g,0}$) this will be a 2-dimensional disc (resp. sphere), from which g pairwise disjoint discs have been removed, and then antipodal points have been identified on each of the resulting boundary components, or equivalently: Möbius bands have been sewn in the place of the removed discs. In Figure 1 the interiors of the removed discs are shaded and numbered from 1 to g. For every nonempty subset $I \subseteq \{1, 2, \ldots, g\}$ let γ_I denote the simple closed

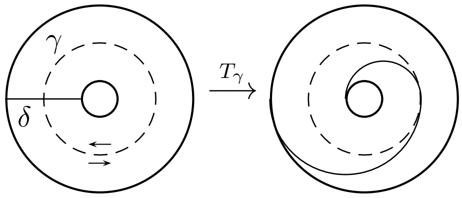

FIGURE 2. Dehn twist about a two-sided curve γ.

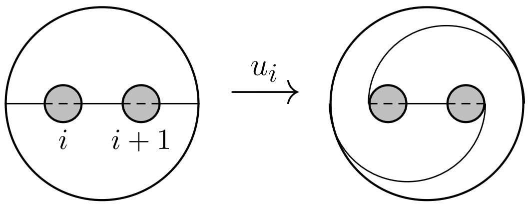

FIGURE 3. Crosscap transposition u_i.

curve on N shown in Figure 1. Note that this curve is one-sided if I has odd cardinality, and two-sided otherwise. With every two-sided simple closed curve γ on N one can associate a Dehn twist about γ, that is an isotopy class of a homeomorphism defined as follows. Choose an oriented closed regular neighbourhood $A \subset N$ of the curve γ, which we identify we the standard annulus $S^1 \times [0,1]$ (Fig. 2). Dehn twist T_γ is equal to the identity outside A, and its action on A is as shown in Figure 2: the interval δ is transformed into the spiral arc, according to the formula

$$T_\gamma(x) = \begin{cases} x & \text{for } x \notin A \\ (e^{2i\pi(\theta+r)}, r) & \text{for } x = (e^{2i\pi\theta}, r) \in A = S^1 \times [0,1]. \end{cases}$$

For $I \subseteq \{1, 2, \ldots, g\}$ of even cardinality we denote by T_I Dehn twist about γ_I in the direction indicated by the arrows in Figure 1. We also set:

$a_i = T_{\{i,i+1\}}$ for $i = 1, 2, \ldots, g-1$;
$b_j = T_{\{1,2,\ldots,2j+2\}}$ for $1 \leq j \leq (g-2)/2$.

For $i = 1, 2, \ldots, g-1$ we define a homeomorphism u_i swapping two consecutive Möbius bands as shown in Figure 3 and equal to the identity outside a one-holed Klein bottle containing these bands. The isotopy class of u_i is denoted by the same symbol and called crosscap transposition. Now we are ready to state the main results of the paper [H5].

Twierdzenie 1 (Paris-Szepietowski [H5, Theorem 3.5]). *For $g \geq 3$ the group $\mathcal{M}(N_{g,1})$ admits a presentation with generators u_i, a_i for $1 \leq i \leq g-1$, b_j for $0 \leq j \leq (g-2)/2$ and relations:*

(A1) $a_i a_j = a_j a_i$ *for* $|i-j| > 1$,

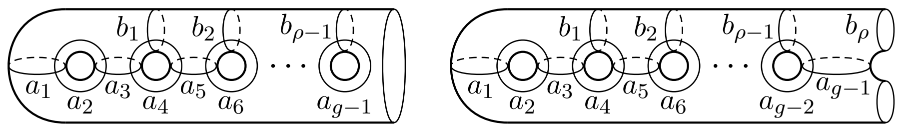

FIGURE 4. The curves on an orientable subsurface of genus $\rho = \lfloor \frac{g-1}{2} \rfloor$ defining the generators a_i, b_j.

(A2) $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ *for* $1 \leq i \leq g-2$,
(A3) $a_i b_1 = b_1 a_i$ *for* $i \neq 4$ *if* $g \geq 4$,
(A4) $b_1 a_4 b_1 = a_4 b_1 a_4$ *if* $g \geq 5$,
(A5) $(a_2 a_3 a_4 b_1)^{10} = (a_1 a_2 a_3 a_4 b_1)^6$ *if* $g \geq 5$,
(A6) $(a_2 a_3 a_4 a_5 a_6 b_1)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b_1)^9$ *if* $g \geq 7$,
(A7) $b_0 = a_1$,
(A8) $b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$
for $2 \leq 2i \leq g-4$,
(A9) $b_{\frac{g-2}{2}} a_{g-5} = a_{g-5} b_{\frac{g-2}{2}}$ *if* g *is even and* $g > 6$,
(A10) $b_2 b_1 = b_1 b_2$ *if* $g = 6$.

(B1) $u_i u_j = u_j u_i$ *for* $|i-j| > 1$,
(B2) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ *for* $i = 1, \ldots, g-2$.

(C1) $a_1 u_i = u_i a_1$ *for* $i = 3, \ldots, g-1$,
(C2) $a_i u_{i+1} u_i = u_{i+1} u_i a_{i+1}$ *for* $i = 1, \ldots, g-2$,
(C3) $a_{i+1} u_i u_{i+1} = u_i u_{i+1} a_i$ *for* $i = 1, \ldots, g-2$,
(C4) $a_1 u_1 a_1 = u_1$,
(C5) $u_2 a_1 a_2 u_1 = a_1 a_2$,
(C6) $(u_3 b_1)^2 = (a_1 a_2 a_3)^2 (u_1 u_2 u_3)^2$ *if* $g \geq 4$,
(C7) $u_5 b_1 = b_1 u_5$ *if* $g \geq 6$,
(C8) $a_4 u_4 (a_4 a_3 a_2 a_1 u_1 u_2 u_3 u_4) b_1 = b_1 a_4 u_4$ *if* $g \geq 5$.

Dehn twists a_i, b_j are defined by curves lying on an orientable subsurface homeomorphic to $S_{\rho,r}$, where $r \in \{1,2\}$ and $g = 2\rho + r$ (Fig. 4). These generators, together with relations (A1-A10) constitute a presentation of the group $\mathcal{M}(S_{\rho,r})$ [H5, Theorem 3.1]. If g is odd, then there are no relations (A9) and (A10), and one can remove from the presentation the generators b_j for $j = 0$ and $j > 1$ and relations (A7, A8). The remaining generators a_i, $i = 1, \ldots, g-1$ and b_1 together with relations (A1-A6) constitute the presentation of $\mathcal{M}(S_{\rho,1})$ found by Matsumoto [68]. If g is even, then one could also rule out b_j for $j \neq 1$. But then in (A9, A10) $b_{\frac{g-2}{2}}$ would have to be replaced by an expression in terms of the generators a_i and b_1. Finding such an explicit expression would considerably simplify our presentation.

The generators u_i, $i = 1, \ldots, g-1$ together with relations (B1, B2) constitute the well known presentation of the braid group B_g. Thus Theorem 1 says that $\mathcal{M}(N_{g,1})$ is isomorphic to the quotient of the free product $\mathcal{M}(S_{\rho,r}) * B_g$ by the relations (C1-C8). To obtain a presentation of $\mathcal{M}(N_{g,0})$ we need to add three more relations.

Twierdzenie 2 (Paris-Szepietowski [H5, Theorem 3.6]). *For $g \geq 4$ the group $\mathcal{M}(N_{g,0})$ is isomorphic to the quotient group obtained by dividing $\mathcal{M}(N_{g,1})$, with the presentation given in Theorem 1, by the relations:*

(B3) $(u_1 u_2 \cdots u_{g-1})^g = 1$,
(B4) $(u_1 u_2 \cdots u_{g-2})^{g-1} = 1$.
(D) $a_1(a_2 a_3 \cdots a_{g-1} u_{g-1} \cdots u_3 u_2) a_1 = a_2 a_3 \cdots a_{g-1} u_{g-1} \cdots u_3 u_2$.

By setting $g = 4$ in Theorem 2 we obtain a presentation of the group $\mathcal{M}(N_{4,0})$ different from that given in [H1, Theorem 2.1]. In [H5, Section 4] we show that these presentations are equivalent, thus performing the base step of the inductive proof of Theorem 2. Thus we can say that the paper [H1] contains a part of the proof of Theorem 2.

The proof of Theorems 1 and 2 are inductive with respect to the genus g, with Theorem 1 being proved under the assumption that Theorem 2 holds. The proof of Theorem 2 uses a theorem of K.S. Brown [16] which allows for computation of a finite presentation of a group acting on a simply-connected CW-complex X by permuting its cells, provided that:

- the stabilizer of each vertex of X is finitely presented;
- the stabilizer of each edge of X is finitely generated;
- the number of orbits of cells of dimension ≤ 2 is finite.

We apply Brown's theorem to the action of $\mathcal{M}(N)$, where $N = N_{g,0}$, $g \geq 4$, on the ordered complex of curves $\mathcal{C}^{\mathrm{ord}}(N)$ defined in [6] similarly as Harvey's curve complex. Two ordered k-tuples of pairwise disjoint and unisotopic simple closed curves on N, $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ and $(\gamma'_1, \gamma'_2, \ldots, \gamma'_k)$, are equivalent if γ_i and γ'_i are isotopic (as unoriented curves) for $i = 1, \ldots, k$. Equivalence classes of such k-tuples are $(k-1)$-simplices of the complex $\mathcal{C}^{\mathrm{ord}}(N)$. Obtaining a presentation of $\mathcal{M}(N)$ by using its action on $\mathcal{C}^{\mathrm{ord}}(N)$ requires a calculation of presentations of the stabilizers of vertices, choosing one representative from each orbit of vertices. The stabilizer $\mathrm{Stab}[\gamma]$ of a vertex $[\gamma]$ is very close to the mapping class group of the compact surface N_γ obtained by cutting N along the curve γ. In particular, one can easily obtain a presentation of $\mathrm{Stab}[\gamma]$ from a presentation of $\mathcal{M}(N_\gamma)$, which can in turn be computed recursively, as N_γ has smaller genus than N. The situation is complicated by the fact that N_γ has nonempty boundary, in contrast to N.

In [P4] I proposed an algorithm, based on the above-mentioned Brown's theorem, of computing a finite presentation of $\mathcal{M}(N)$. The presentation resulting from this algorithm is finite but enormous; it contains recursively computed presentations of stabilizers of vertices of the complex $\mathcal{C}^{\mathrm{ord}}(N)$, and many relations corresponding to cells of dimension 1 and 2. To obtain an explicit presentation of $\mathcal{M}(N)$ with reasonable numbers of generators and relations, we need to apply this algorithm in a subtle way, so that the presentations obtained in the intermediate steps are not too big. In [H1] this was achieved for $g = 4$, and the ultimate goal, that is an explicit finite presentation of $\mathcal{M}(N_g)$ for all g, was reached in [H5]. Thanks to having the case $g = 4$ solved in the earlier paper [H1], in [H5] we could use the ground-breaking idea of replacing the complex $\mathcal{C}^{\mathrm{ord}}(N)$ by its subcomplex build only from nonseparating curves, which is simply-connected for $g \geq 5$. In the case $g \geq 7$ we used an even smaller subcomplex, which considerably reduced the presentation resulting from Brown's theorem.

Starting from the presentation of $\mathcal{M}(N_{g,0})$ one can inductively calculate presentations of $\mathcal{M}(N_{g,n}, P)$ for arbitrary n and P by a method based on the Birman exact sequence, as in the paper [60] in the case of orientable surfaces. Finding such a presentation in the general case is an interesting research challenge.

From the presentations given in Theorems 1 and 2 one can quite easily rule out the generators u_i for $i > 1$. This was done by Stukow [78], who obtained in this way presentations of $\mathcal{M}(N_{g,1})$ and $\mathcal{M}(N_{g,0})$ with smaller numbers of generators and relations, and by using these presentations he computed the first homology group of $\mathcal{M}(N_{g,n})$ with coefficients in $H_1(N_{g,n}; \mathbb{Z})$ for $n \leq 1$ [79]. Recently, Omori posted to the arXiv repository an interesting preprint [72], providing infinite presentations of the groups $\mathcal{M}(N_{g,1})$ and $\mathcal{M}(N_{g,0})$ with very simple relations. Generators in this presentations are all Dehn twists and all Y-homeomorphisms (also called *crosscap slides* and described below in Section 2.4). The proof of the main result of [72] uses Stukow's presentation [78], and thus, indirectly, also Theorems 1 and 2.

It is worth adding that a presentation of $\mathcal{M}(N_{g,n})$ with only Dehn twists as generators is impossible. Indeed, the subgroup of $\mathcal{M}(N_{g,n})$ generated by all Dehn twists has index 2 [62, 76].

2.3. **Linear representations and other homomorphisms.** [H4]

The action of the group $\mathcal{M}(S_{g,n})$ on $H_1(S_g, \mathbb{Z})$ preserves the algebraic intersection pairing, which is a symplectic form. The induced surjective homomorphism

$$\Phi \colon \mathcal{M}(S_{g,n}) \to \mathrm{Sp}(2g, \mathbb{Z}),$$

called standard symplectic representation, is an important tool in the study of the mapping class group of an orientable surface. In recent years, J. Franks, M. Handel and M. Korkmaz [23, 57, 58] proved that for $g \geq 3$ the smallest degree of a nontrivial representation $\mathcal{M}(S_{g,n}) \to \mathrm{GL}(m, \mathbb{C})$ is $m = 2g$, and that the standard symplectic representation is the unique, up to conjugation in $\mathbb{C}$, complex representation of $\mathcal{M}(S_{g,n})$ of degree $2g$. In the paper [H4] I proved analogous results for the mapping class group of a non-orientable surface.

We say that two group homomorphisms f_1, f_2 from G to H are conjugate if there exists $y \in H$ such that $f_1(x) = y f_2(x) y^{-1}$ for all $x \in G$. The image of a homomorphism f is denoted by $\mathrm{Im}(f)$.

Let us fix a double covering $P \colon S_{g-1} \to N_g$. By the theorem of Birman and Chillingworth [9], $\mathcal{M}(N_g)$ is isomorphic to the subgroup of $\mathcal{M}(S_{g-1})$ consisting of the orientation preserving lifts of homeomorphisms of N_g. Thus we have an action of $\mathcal{M}(N_g)$ on $H_1(S_{g-1}, \mathbb{Z})$. We denote by K_g the kernel of the homomorphism

$P_*\colon H_1(S_{g-1},\mathbb{Z}) \to H_1(N_g,\mathbb{Z})/\mathbb{Z}_2$ induced by the covering P, where $\mathbb{Z}_2$ denotes the torsion subgroup of $H_1(N_g,\mathbb{Z})$. The group K_g is invariant under the action of $\mathcal{M}(N_g)$ on $H_1(S_{g-1},\mathbb{Z})$. Furthermore, K_g and $H_1(S_{g-1},\mathbb{Z})/K_g$ are free $\mathbb{Z}$-modules of rank $g-1$, and hence we obtain two representations of $\mathcal{M}(N_g)$ of rank $g-1$

$$\Psi_1\colon \mathcal{M}(N_g) \to \mathrm{GL}(K_g), \qquad \Psi_2\colon \mathcal{M}(N_g) \to \mathrm{GL}(H_1(S_{g-1},\mathbb{Z})/K_g),$$

which, after fixing bases, will be treated as homomorphisms to $\mathrm{GL}(g-1,\mathbb{C})$. It turns out that they are not conjugate, although $\ker\Psi_1 = \ker\Psi_2$ [H4, Lemma 4.1]. The first result of the paper [H4] says, that $g-1$ is the smallest degree of a nontrivial (nonabelian) representation of $\mathcal{M}(N_g)$.

Twierdzenie 3 (Szepietowski [H4, Theorem 1.3])**.** *Let $n \le 1$, $g \ge 5$, $m \le g-2$ and suppose that $f\colon \mathcal{M}(N_{g,n}) \to \mathrm{GL}(m,\mathbb{C})$ is a nontrivial homomorphism. Then $\mathrm{Im}(f)$ is isomorphic either to $\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_2$, and the latter case is possible only for $g=5$ or 6.*

The above result was proved by Korkmaz in [57] under the additional assumption that $m \le g-3$ if g is even. The novelty of Theorem 3 consist in the fact that it also covers the case $m = g-2$ for even g. As an application of Theorem 3 I proved the following result, which solves Problem 3.3 in [56]

Twierdzenie 4 (Szepietowski [H4, Theorem 1.4])**.** *Suppose that $g \ge 5$, $h < g$ and $f\colon \mathcal{M}(N_g) \to \mathcal{M}(N_h)$ is a nontrivial homomorphism. Then $\mathrm{Im}(f)$ is as in Theorem 3.*

The analogous theorem for mapping class groups of orientable surfaces was proved by Harvey and Korkmaz [36]. Theorems 3 and 4 both fail for $g=4$, as I showed that there is a homomorphism from $\mathcal{M}(N_4)$ to $\mathcal{M}(N_3) \cong \mathrm{GL}(2,\mathbb{Z})$, whose image is isomorphic to the infinite dihedral group [H4, Corollary 6.2]. To construct such a homomorphism I used the presentation of the group $\mathcal{M}(N_4)$ from the papers [H1, H5].

Suppose that $g \ge 7$. Then the abelianization of $\mathcal{M}(N_g)$ is isomorphic to $\mathbb{Z}_2$ [52]. We denote by $\mathrm{ab}\colon \mathcal{M}(N_g) \to \mathbb{Z}_2$ the canonical projection and for $i=1,2$ we define $\Psi'_i\colon \mathcal{M}(N_g) \to \mathrm{GL}(g-1,\mathbb{C})$ by the formula $\Psi'_i(x) = (-1)^{\mathrm{ab}(x)}\Psi_i(x)$ for $x \in \mathcal{M}(N_g)$. The next result of the paper [H4] is the following.

Twierdzenie 5 (Szepietowski [H4, Theorem 1.5])**.** *Let $g \ge 7$, $g \ne 8$ and suppose that $f\colon \mathcal{M}(N_g) \to \mathrm{GL}(g-1,\mathbb{C})$ is a nontrivial homomorphism. Then either $\mathrm{Im}(f) \cong \mathbb{Z}_2$, or f is conjugate to one of the homomorphisms Ψ_1, Ψ'_1, Ψ_2, Ψ'_2.*

For $g=8$ I proved analogous theorem [H4, Theorem 1.6]. In this case we have an additional homomorphism $\mathcal{M}(N_8) \to \mathrm{GL}(7,\mathbb{C})$ related to the fact that there is an epimorphism from $\mathcal{M}(N_8)$ onto $\mathrm{Sp}(6,\mathbb{Z}_2)$, and the last group admits an irreducible representation in $\mathrm{GL}(7,\mathbb{C})$.

2.4. **Level 2 mapping class group.** [H2, H3]

By composing the standard symplectic representation of the group $\mathcal{M}(S_g)$ with the homomorphism of reduction modulo m, for some natural $m \ge 2$, we obtain a surjective representation $\mathcal{M}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z}_m)$, whose kernel is denoted by $\Gamma_m(S_g)$ and called level m mapping class group of the surface S_g. The group $\Gamma_m(S_g)$ may also be described as the group of isotopy classes of homeomorphisms of S_g inducing the identity on $H_1(S_g, \mathbb{Z}_m)$. Summarising, we have an exact sequence

$$1 \to \Gamma_m(S_g) \to \mathcal{M}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z}_m) \to 1.$$

The groups $\Gamma_m(S_g)$ have been intensively studied, among others by Hain [29] and Ivanov [45], and from more recent results it is worth mentioning the computation of their abelianization [74, 75].

In the case of a nonorientable surface N_g, the algebraic intersection pairing on $H_1(N_g, \mathbb{Z})$ is defined only modulo 2. For this reason it is very natural to consider the action of $\mathcal{M}(N_g)$ on $H_1(N_g, \mathbb{Z}_2)$ and its kernel $\Gamma_2(N_g)$. The group of automorphisms of $H_1(N_g, \mathbb{Z}_2)$ preserving the algebraic intersection form is denoted, after Korkmaz [52], by $\mathrm{Iso}(H_1(N_g, \mathbb{Z}_2))$. By fixing the standard basis of $H_1(N_g, \mathbb{Z}_2)$ we have the isomorphism

$$\mathrm{Iso}(H_1(N_g, \mathbb{Z}_2)) \cong \{A \in \mathrm{GL}(g, \mathbb{Z}_2) \mid AA^t = I\}.$$

McCarthy and Pinkall [69], and also Gadgil and Pancholi [24] proved that the mapping $\mathcal{M}(N_g) \to \mathrm{Iso}(H_1(N_g, \mathbb{Z}_2))$ is a surjection. We thus have an exact sequence

$$1 \to \Gamma_2(N_g) \to \mathcal{M}(N_g) \to \mathrm{Iso}(H_1(N_g, \mathbb{Z}_2)) \to 1.$$

The papers [H2, H3] are devoted to the group $\Gamma_2(N_g)$. For the formulation of their results, the notion of a Y-homeomorphism is needed.

In contrast to $M(S_g)$, the group $\mathcal{M}(N_g)$ is not generated by Dehn twists. This was proved by Lickorish [61], who gave the first example of an element of $\mathcal{M}(N_g)$ which is not a product of Dehn twists, namely the Y-homeomorphism, also called crosscap slide. Let $g \geq 2$ and suppose that α and β are simple closed curves on N_g, intersecting in one point, and such that α is one-sided and β two-sided. Let $K \subset N_g$ be a regular neighbourhood of $\alpha \cup \beta$, homeomorphic to a one-holed Klein bottle. Denote by M a regular neighbourhood of α, which is a Möbius band. The Y-homeomorphism $Y_{\alpha,\beta}$ may be described as the effect of pushing M once along β keeping each point on the boundary of K fixed, and equal to the identity outside K (Fig.5).

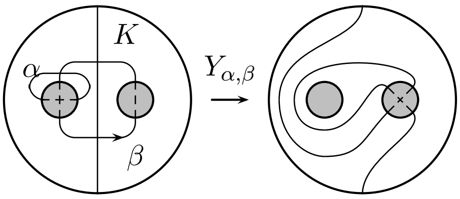

FIGURE 5. Y-homeomorphism or crosscap slide.

Lickorish proved that for $g \geq 2$ the group $\mathcal{M}(N_g)$ is generated by Dehn twists and one Y-homeomorphism, and the subgroup generated by all Dehn twists has index 2 [61, 62]. We denote by $\mathcal{Y}(N_g)$ the subgroup of $\mathcal{M}(N_g)$ generated by all Y-homeomorphisms. It is easy to check that every Y-homeomorphism induces the identity on $H_1(N_g, \mathbb{Z}_2)$, and hence $\mathcal{Y}(N_g) \subseteq \Gamma_2(N_g)$. In the paper [H2] I proved the equality $\mathcal{Y}(N_g) = \Gamma_2(N_g)$.

Twierdzenie 6 (Szepietowski [H2, Theorem 5.5]). *Let $g \geq 2$. An element $f \in \mathcal{M}(N_g)$ induces the identity on $H_1(N_g, \mathbb{Z}_2)$ if and only if f is a product of Y-homeomorphisms.*

In particular, $\mathcal{Y}(N_g)$ is a proper subgroup of $\mathcal{M}(N_g)$ of finite index. For $I, J \subseteq \{1, 2, \ldots, g\}$ we denote Y_{γ_I, γ_J} by $Y_{I;J}$, where γ_I, γ_J are the curves from Figure 1, provided that these curves satisfy the assumptions of the definition of a Y-homeomorphism. I proved that $\mathcal{Y}(N_g)$ is the normal closure in $\mathcal{M}(N_g)$ of one Y-homeomorphism $Y_{\{1\};\{1,2\}}$ [H2, Lemma 3.6], which is the product of two involutions belonging to $\mathcal{Y}(N_g)$. Thus I proved the following theorem.

Twierdzenie 7 (Szepietowski [H2, Theorem 3.7 i Corollary 5.7]). *For $g \geq 2$ the group $\Gamma_2(N_g)$ is generated by involutions.*

It follows from the last theorem that the abelianization of $\Gamma_2(N_g)$ is a $\mathbb{Z}_2$-module.

Since $\mathcal{M}(N_g)$ is finitely generated, so is $\Gamma_2(N_g)$ as a subgroup of finite index. Therefore, it is a natural problem to find a finite generating set for $\Gamma_2(N_g)$. I solved this problem in the paper [H3].

Twierdzenie 8 (Szepietowski [H3, Theorem 3.2]). *For $g \geq 3$, the group $\Gamma_2(N_g)$ is generated by the following elements:*

(1) $Y_{\{i\};\{i,j\}}$ for $i \in \{1, 2, \ldots, g-1\}$, $j \in \{1, 2, \ldots, g\}$, $i \neq j$;
(2) $Y_{\{i,j,k\};\{i,j,k,l\}}$ for $i < j < k < l$, if $g \geq 4$.

Let us add, for completeness, that $\Gamma_2(N_1) = \mathcal{M}(N_1) = \{1\}$ and $\Gamma_2(N_2) \cong \mathbb{Z}_2$.

In Theorem 8, every generator $Y_{\{i,j,k\};\{i,j,k,l\}}$ of type (2) can be replaced by $T^2_{\{i,j,k,l\}}$, where $T_{\{i,j,k,l\}}$ is Dehn twist about $\gamma_{\{i,j,k,l\}}$ [H3, Remark 3.9]. Note that there are $(g-1)^2$ generators of type (1) and $\binom{g}{4}$ generators of type (2). In the final section of the paper [H3] I proved that the number of generators of $\Gamma_2(N_g)$ from Theorem 8 is minimal for $g = 3$ and 4. The action of $\mathcal{M}(N_3)$ on $H_1(N_3, \mathbb{Z})$ induces an isomorphism $\mathcal{M}(N_3) \to \mathrm{GL}(2, \mathbb{Z})$, which maps $\Gamma_2(N_3)$ on the level 2 principal congruence subgroup of $\mathrm{GL}(2, \mathbb{Z})$ [H3, Corollary 4.2]. The next theorem says that the number of generators of $\Gamma_2(N_4)$ from Theorem 8 is equal to the rank of the abelianization of this group, and hence is minimal.

Twierdzenie 9 (Szepietowski [H3, Theorem 4.3]). *The group $H_1(\Gamma_2(N_4), \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2^{10}$.*

The proof of Theorem 9 uses Theorems 7 and 8, and also the presentation of $\mathcal{M}(N_4)$ from the paper [H1]. For $g > 4$ the generating set of $\Gamma_2(N_g)$ from Theorem 8 is not minimal. Hirose and Sato [41] showed that it contains a subset of cardinality $\binom{g+1}{3}$, which also generates $\Gamma_2(N_g)$, and then they proved that $H_1(\Gamma_2(N_g), \mathbb{Z})$

has rank $\binom{g+1}{3}$, which is a generalisation of the above Theorem 9. For their computation of the abelianization of $\Gamma_2(N_g)$ Hirose and Sato use my Theorems 7 and 8.

The paper [H2] contains an important construction of the homomorphism *crosscap pushing map*

$$\psi\colon \pi_1(N_{g-1}, x_0) \to \mathcal{M}(N_g),$$

where N_{g-1} is obtained by removing from N_g a Möbius band, and gluing a disc with a distinguished point x_0 in its place. If $\alpha \in \pi_1(N_{g-1}, x_0)$ is a homotopy class represented by a simple closed curve then $\psi(\alpha)$ is either a Y-homeomorphism if α is one-sided, or a product of two Dehn twists if α is two-sided. This allows for obtaining relations in $\mathcal{M}(N_g)$ of the form

$$\psi(\alpha\beta) = \psi(\alpha)\psi(\beta), \tag{1}$$

where on each side of the equality there are Y-homeomorphisms or Dehn twists, provided that α, β and $\alpha\beta$ are represented by simple curves (here the product $\alpha\beta$ in $\pi_1(N_{g-1}, x_0)$ means first β, and then α). Certain relations appearing in the finite presentations of the groups $\mathcal{M}(N_g)$ and $\mathcal{M}(N_{g,1})$ found in the papers [H5] and [78] were obtained in this way, by using the crosscap pushing map. Furthermore, (1) is one on the defining relations in Omori's infinite presentation [72]. The crosscap pushing map ψ is a basic tool for studying Y-homeomorphisms, used in the papers [H2, H3], and also in works of other authors, including [42] and the above-mentioned papers [72, 78]. I believe that this tool has a big potential, as the study of Y-homeomorphisms is an important part of the theory of the mapping class group of a nonorientable surface.

The group $\Gamma_2(N_g)$ may be seen as certain approximation of the Torelli subgroup $\mathcal{I}(N_g)$ consisting of the elements of $\mathcal{M}(N_g)$ inducing the identity on $H_1(N_g, \mathbb{Z})$. On the one hand this approximation is very inaccurate as $\mathcal{I}(N_g)$ is a subgroup of $\Gamma_2(N_g)$ of infinite index. On the other hand however, the finite generating set of $\Gamma_2(N_g)$ appearing in Theorem 8 and reduced in [41] is one of the ingredients of the proof of the main theorem of the paper [42], in which Hirose and Kobayashi found certain infinite generating set of $\mathcal{I}(N_g)$. Their result is analogous to the classical theorem of Powell [73] about generators of the Torelli group of an orientable surface. It is worth adding that, as of now, no finite generating set of $\mathcal{I}(N_g)$ is know.

Theorems 6 and 8 have also been used in the proof of the main theorem of the paper [40] providing a necessary and sufficient condition for a homeomorphism of a nonorientable surface, embedded in a certain standard way in the 4-sphere S^4, to extend to a homeomorphism of S^4.

3. Other scientific achievements

3.1. Before Ph.D.

[P1] B. Szepietowski, Mapping class group of a non-orientable surface and moduli space of Klein surfaces, *Comptes Rendus de l'Academie des Sciences*, Paris, Ser. I 335 (2002) 1053–1056.

[P2] B. Szepietowski, Involutions in mapping class groups of non-orientable surfaces, *Collectanea Mathematica* 55, 3 (2004), 253–260.

[P3] B. Szepietowski, The mapping class group of a nonorientable surface is generated by three elements and by four involutions, *Geometriae Dedicata* 117 (2006), 1–9.

[P4] B. Szepietowski, A presentation for the mapping class group of a non-orientable surface from the action on the complex of curves, *Osaka Journal of Mathematics* 45 (2008), 283–326.

3.2. After Ph.D.

[P5] B. Szepietowski, On the commutator length of a Dehn twist, *Comptes Rendus Mathematuque* 348 (2010), 923–926.

[P6] E. Bujalance, F. J. Cirre, M. D. E. Conder, B. Szepietowski, Finite group actions on bordered surfaces of small genus, *Journal of Pure and Applied Algebra* 214 (2010), 2165–2185.

[P7] B. Szepietowski, Embedding the braid group in mapping class groups, *Publicacions Matematiques* 54 (2010), 359–368.

[P8] B. Szepietowski, Counting pseudo-Anosov elements in the mapping class group of the three-punctured projective plane, *Turkish Journal of Mathematics* 38 (2014) 524–534.

[P9] B. Szepietowski, On finite index subgroups of the mapping class group of a nonorientable surface, *Glasnik Matematički* 49 (2014) 337–350.

[P10] E. Bujalance, J. J. Etayo, E. Martínez, B. Szepietowski, On the connectedness of the branch loci of non-orientable unbordered Klein surfaces of low genus, *Glasgow Mathematical Journal* 57 (2015), 211–230.

[P11] G. Gromadzki, B. Szepietowski, On topological type of periodic self-homeomorphisms of closed non-orientable surfaces, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A, Matemáticas*, online first 2015, 18 pp. DOI: 10.1007/s13398-015-0234-6

[P12] G. Gromadzki, B. Szepietowski, X. Zhao, On classification of cyclic orientation-reversing actions of big order on closed surfaces, *Journal of Pure and Applied Algebra*, 220 (2016), 465-481.

[P13] F. Atalan, B. Szepietowski, Automorphisms of the mapping class group of a nonorientable surface, preprint 2014, ArXiv:1403.2774. Submitted for publication.

In preparation:

[P14] G. Gromadzki, S. Hirose, B. Szepietowski, On topological classification of finite cyclic actions on bordered surfaces, preprint 2015.

Below I describe the main results of the above papers, starting from those devoted strictly to mapping class groups. Then I will describe the papers concerning other subjects, written in collaboration with other mathematicians, in which I managed to use my experience from the study of the mapping class group. These are 4 papers on topological classification of actions of finite groups on surfaces [P6, P11, P12, P14] and one paper about the connectivity of the branch locus of the moduli space of Klein surfaces [P10].

3.3. **Finite index subgroups of the mapping class group of a nonorientable surface.** [P9]

By Grossman's theorem [28], the group $\mathcal{M}(S_{g,n})$ is residually finite, and since $\mathcal{M}(N_{g,n})$ is isomorphic to a subgroup of $\mathcal{M}(S_{g-1,2n})$, it is residually finite as well. This means that mapping class groups have a rich supply of finite index subgroups. It is worth remarking that to every such subgroup corresponds certain finite degree covering of the appropriate moduli space. On the other hand, A. J. Berrick, V. Gebhardt and L. Paris [8] proved that for $g \geq 3$ the minimum index of a proper subgroup of $\mathcal{M}(S_{g,n})$ is $2^{g-1}(2^g-1)$. More specifically, it is proved in [8] that $\mathcal{M}(S_{g,n})$ contains a unique subgroup of index $m_g^- = 2^{g-1}(2^g-1)$ up to conjugation, a unique subgroup of index $m_g^+ = 2^{g-1}(2^g+1)$ up to conjugation, and all other proper subgroups of $\mathcal{M}(S_{g,n})$ have index strictly greater than m_g^+ (and at least $5m_g^-$ if $g \geq 4$).

For $g \geq 2$ the minimum index of a proper subgroup of $\mathcal{M}(N_{g,n})$ is 2, and if $g \geq 7$ then the subgroup generated by all Dehn twists, denoted by $\mathcal{T}(N_{g,n})$, is the unique subgroup of $\mathcal{M}(N_{g,n})$ of index 2. Suppose that $g \geq 7$, $n \in \{0,1\}$ and set $h = \lfloor (g-1)/2 \rfloor$. Let G denote either $\mathcal{M}(N_{g,n})$ or $\mathcal{T}(N_{g,n})$. In [P9, Theorem 1.1] I proved that G contains a unique subgroup of index $m_h^- = 2^{h-1}(2^h-1)$ up to conjugation, a unique subgroup of index $m_h^+ = 2^{h-1}(2^h+1)$ up to conjugation, and all other proper subgroups of G have index strictly greater than m_h^+ (and at least $5m_h^-$ if $h \geq 4$). In particular, the minimum index of a proper subgroup of $\mathcal{T}(N_{g,n})$ is m_h^-.

For $2 \leq g \leq 6$ the minimum index of a proper subgroup of $\mathcal{T}(N_{g,n})$ is 2. For $g \in \{5,6\}$ I proved [P9, Theorem 4.1], that $\mathcal{T}(N_{g,n})$ contains a unique subgroup of index 2, two subgroups of index $m_2^- = 6$ and one subgroup of index $m_2^+ = 10$ up to conjugation, and all other proper subgroups of $\mathcal{T}(N_{g,n})$ gave index greater than 10. Since the abelianization of $\mathcal{T}(N_{4,0})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ [76], every positive integer is the index of some subgroup of $\mathcal{T}(N_{4,n})$.

3.4. **Embeddings of the braid group in mapping class groups.** [P7]

When two two-sided simple closed curves α, β on a surface F do not intersect, then the corresponding Dehn twists commute: $T_\alpha T_\beta = T_\beta T_\alpha$; whereas if α and β intersect in one point, then the twits satisfy in $\mathcal{M}(F)$ the braid relation: $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$ (provided that the directions of the twists agree at the intersection point. Thus, to each chain $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ of two-sided simple closed curves on F, where $\alpha_i \cap \alpha_j = \emptyset$ for $|i-j| > 1$ and α_i intersects α_{i+1} in one point for $i = 1, 2, \ldots, n-2$, corresponds a homomorphism from the braid group B_n on n strands to the mapping class group $\mathcal{M}(F)$. Such a homomorphism is in general injective. The paper [P7] was motivated by a question of B. Wajnryb [87] about existence of „nongeometric" embeddings $B_n \to \mathcal{M}(F)$, such that the images of the standard generators of B_n are not Dehn twists. In the paper [P7] I proved that mapping the standard generators of B_g on the crosscap transpositions u_i (Fig. 3), $i = 1, \dots, g-1$ defines an embedding

$$\varphi\colon B_g \to \mathcal{M}(N_{g,1}).$$

In the same paper I extended the theorem of Birman and Chillingworth to surfaces with boundary by proving that $\mathcal{M}(N_{g,n})$ is isomorphic to a subgroup of $\mathcal{M}(S_{g-1,2n})$, which allowed for defining

$$\psi\colon B_g \to \mathcal{M}(S_{g-1,2})$$

by lifting the u_i from $N_{g,1}$ to the double cover $S_{g-1,2}$. Both embeddings φ and ψ have the property that the images of the standard generators of B_n are not Dehn twists. Bödigheimer and Tillmann [11] proved that the embedding ψ induces the zero map between the homology groups of positive degrees, as long as the genus of the underlying surface is large enough relative to the degree. Also the standard geometric embeddings have this property, as well as some other nongeometric embeddings of the braid group in the mapping class group of an orientable surface described in [11]. In contrast, the map $\varphi_*\colon H_k(B_g;\mathbb{Z}_2) \to H_k(\mathcal{M}(N_{g,1});\mathbb{Z}_2)$ induced by the embedding φ is injective for $g \geq 7$ and $0 < k \leq g/3$ [11].

3.5. **Dehn twist as a commutator.** [P5]

The subgroup of a group G generated by all commutators $[a,b] = aba^{-1}b^{-1}$, $a, b \in G$ is denoted by $[G,G]$. For $x \in [G,G]$ let $cl_G(x)$ denote the smallest number k such that x is a product of k commutators, and let $scl_G(x)$ be the limit

$$scl_G(x) = \lim_{n\to\infty} \frac{cl(x^n)}{n}.$$

The numbers $cl_g(x)$ and $scl_G(x)$ are called respectively the commutator length and the stable commutator length of the element x in the group G.

Suppose that S is a closed orientable surface of genus $g \geq 3$. The mapping class group $\mathcal{M}(S)$ is perfect, i.e. $[\mathcal{M}(S), \mathcal{M}(S)] = \mathcal{M}(S)$ [73]. Let α be a simple closed curve on S, not contractible to a point, and let T_α be Dehn twist about α. Then $cl_{\mathcal{M}(S)}(T_\alpha) = 2$ [59] and $scl_{\mathcal{M}(S)}(T_\alpha) \geq \frac{1}{18g-6}$ [20, 54]. In particular, the sequence $cl_{\mathcal{M}(S)}(T_\alpha^n)$, $n \in \mathbb{Z}$ is unbounded. The extended mapping class group $\mathcal{M}^\diamond(S)$ is defined as the group of isotopy classes of all homeomorphisms of S, including those reversing orientation. In the paper [P5] I proved that T_α^n is equal to a single commutator of elements of $\mathcal{M}^\diamond(S)$ for every $n \in \mathbb{Z}$. Hence $cl_{\mathcal{M}^\diamond(S)}(T_\alpha^n) = 1$ and $scl_{\mathcal{M}^\diamond(S)}(T_\alpha) = 0$.

Suppose that N is a closed nonorientable surface of genus $g \geq 7$. Then we have $[\mathcal{M}(N), \mathcal{M}(N)] = \mathcal{T}(N) = [\mathcal{T}(N), \mathcal{T}(N)]$, where $\mathcal{T}(N)$ is the subgroup of $\mathcal{M}(N)$ of index 2 generated by all Dehn twists [52]. In the paper [P5] I proved that $cl_{\mathcal{M}(N)}(T_\alpha^n) = 1$ for every two-sided simple closed curve α on N and all $n \in \mathbb{Z}$, and under certain additional assumptions about α and N also $cl_{\mathcal{T}(N)}(T_\alpha^n) = 1$.

3.6. **Growth function and density of pseudo-Anosov elements in the mapping class group of the projective plane with 3 punctures.** [P8]

A group G with a fixed generating set A can be equipped with a metric called word metric. In this metric, the length of an element x is the minimum number of factors needed to express x as a product of generators from the set A. For any subset X of G we can define a power series, whose coefficient a_n is equal to the number of elements of X of length n. This series is called growth series, and the function it defines is called growth function. Density of the set X is defined as the limit

$$\lim_{n\to\infty} \frac{|B(n) \cap X|}{|B(n)|},$$

where $B(n)$ denotes the set of elements of G of length at most n.

Let N be a nonorientable surface with a finite set P of distinguished points (punctures). The pure mapping class group $\mathcal{PM}(N,P)$ is defined as the group of isotopy classes of homeomorphisms of N fixing every point of P and preserving local orientation in every point of P. In the paper [P8] I consider the group $\mathcal{PM}(N,P)$, where (N,P) is the projective plane with 3 punctures, equipped with the word metric induced by a certain fixed generating set. I computed the growth functions of the sets of reducible and pseudo-Anosov elements. These functions turned out to be rational. I also proved that the set of pseudo-Anosov elements has density 1.

Analogous results were obtained in [2] for the sphere with 4 punctures, and in [81] for the torus. The described results give a partial answer to Question 3.13 and confirm Conjecture 3.15 in [22] in a special case.

3.7. **Other papers devoted to the mapping class group of a nonorientable surface.** [P1-P4,P13]

The paper [P1] contains the main results of my master thesis, whereas the papers [P2, P3, P4] are the core of my Ph.D. thesis, although [P4] appeared two years after my Ph.D.

Let N_g denote a closed nonorientable surface of genus $g \geq 3$. In the paper [P1] I proved that the mapping class group $\mathcal{M}(N_g)$ is generated by involutions. As an important application of this result, I proved simple connectivity of the moduli space $\mathfrak{M}(N_g)$ of Klein surfaces homeomorphic to N_g, following the proof of simple connectivity of the moduli space of Riemann surfaces given by Maclachlan [64]. In [P2] I proved that the group $\mathcal{M}(N_g, P)$, where P is a finite set of distinguished points on N_g, is also generated by involutions. In [P3] I proved that $\mathcal{M}(N_g)$ is generated by three elements, and also is generated by four involutions. The paper [P3] was inspired by the articles [13, 50, 55, 84] containing similar results for the mapping class group of an orientable surface. In [P4] I proposed a recursive algorithm for obtaining a finite presentation of the mapping class group

$\mathcal{M}(N_{g,n})$ by using its action on the curve complex. This algorithm was used in the papers [H1, H5]. In [P4] I found finite presentations of the groups $\mathcal{M}(N_{g,n})$ for $(g, n) \in \{(1, 3), (1, 4), (2, 2), (2, 3), (3, 1)\}$. For these surfaces the curve complex is not simply connected.

In the paper [P13], as yet unpublished, we proved, jointly with F. Atalan, that if N is a closed nonorientable surface of genus $g \geq 5$ with a finite (possibly empty) set of distinguished points P, then every automorphism of the group $\mathcal{M}(N, P)$ is inner. Analogous theorem for the mapping class group of an orientable surface is due to Ivanov [44]. He prove that if S is an orientable surface of genus $g \geq 3$ with a finite set of distinguished points P, then every automorphism of $\mathcal{M}(S, P)$ is induced by a homeomorphism of S, not necessarily orientation preserving one.

3.8. **Topological classification of finite group actions on compact surafces.** [P6, P11, P12, P14].

By an action of a group G on a surface F we understand an embedding of G in Homeo(F), and two such actions are called topologically equivalent if their images are conjugate in Homeo(F). Classification of finite group actions on compact surfaces up to topological equivalence is a classical problem, going back to Nielsen, with a vast literature, especially in the case of orientable surfaces.

In the papers [P6, P11, P12, P14] we use the methods of combinatorial theory of noneuclidean crystallographic groups, NEC groups in short, which are discrete and cocompact subgroups of the group of isometries of the hyperbolic plane $\mathcal{H}$, initiated by Macbeath [63]. An action of a finite group G on a compact surface F of negative Euler characteristic can be realised by an analytic or dianalytic action, with respect to some structure of a Riemann or Klein surface on F. This means that such an action can be defined by a smooth epimorphism $\theta \colon \Lambda \to G$, where Λ is a certain NEC group, and whose kernel is also a NEC group, torsion-free if F is closed, or containing no orientation preserving isometries of finite order if F is a surface with boundary. The point is, that the topology of the action of G is determined by algebraic features of θ and Λ. Thus, in the study of finite group actions we can restrict ourselves to algebra and combinatorics, and forget about the analytic aspects. In this language, two actions of a group G on F are topologically equivalent if and only if the corresponding smooth epimorphisms $\theta_i \colon \Lambda_i \to G$, $i = 1, 2$, fit in the commutative diagram

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\theta_1} & G \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \Lambda_2 & \xrightarrow{\theta_2} & G \end{array} \tag{2}$$

where α and β are certain isomorphisms. To tell if two given smooth epimorphism $\Lambda \to G$ define topologically equivalent actions, we thus need to know the group of automorphisms of the NEC group Λ. At this point we use a close relationship between the group Out(Λ) of outer automorphisms of Λ and appropriately defined mapping class group $\mathcal{M}(\mathcal{H}/\Lambda)$ of the orbifold $\mathcal{H}/\Lambda$. Knowing generators of

$\mathcal{M}(\mathcal{H}/\Lambda)$ we can easily obtain generators of $\mathrm{Out}(\Lambda)$, and if the order of the group G is large enough relative to the genus of the surface, then the groups $\mathcal{M}(\mathcal{H}/\Lambda)$ are $\mathrm{Out}(\Lambda)$ finite, which allows for an effective study of topological equivalence of group actions given by smooth epimorphisms.

The series of papers [P11, P12, P14] is devoted to actions of finite cyclic groups of big order on closed surfaces . At the end of the XIX century Wiman [89] proved that the order of an orientation-preserving automorphism of a Riemann surface of genus $g \geq 2$ is at most $4g + 2$, and Harvey [33] proved that this bound is attained for all $g \geq 2$. Analogous results about the maximum orders of an orientation-preserving periodic homeomorphism and a periodic homeomorphism of a nonorientable surface were obtained in the papers [17, 21, 88]. A natural question is to what extent the order of a periodic homeomorphism of a surface determines its conjugacy class. In the case of orientation-preserving homeomorphisms of S_g it was known that the order determines the conjugacy class, as long as this order and the genus g are large enough [4, 39]. In the papers [P11] and [P12] we consider the analogous problem respectively for homeomorphisms of N_g, $g \geq 3$ and for orientation-reversing homeomorphisms of S_g, $g \geq 2$. In [P11] (respectively [P12]) we computed the numbers of topologically inequivalent actions of a cyclic group $\mathbb{Z}_n$ on N_g (resp. on S_g containing orientation-reversing homeomorphisms), depending on the type of the orbifold $N_g/\mathbb{Z}_n$, for $n > g - 2$ (resp. $S_g/\mathbb{Z}_n$, for $n > 2g - 2$). In particular, we proved that the actions of maximal order are unique up to topological equivalence, with the exception of a non-orientable surface of even genus g, on which we have two different topological types of an action of maximal order $n = 2g$. It worth emphasising that although in the theorems stated in [P11, P12] we give only the numbers of topological types of actions of big order, in the proofs we obtain the corresponding smooth epimorphisms, and thus we obtain their topological classification. The paper [P14], in preparation, contains analogous classification of $\mathbb{Z}_n$-actions on surfaces with boundary, such that $n > p - 2$, where p is the algebraic genus of the surface. In particular, we classify the actions realizing the solutions of the so called minimal genus and maximal order problems for surfaces with boundary, found thirty years ago in [18].

In the paper [P6] we classified, up to topological equivalence, all actions of groups of finite order at least 6 on compact surfaces with boundary of algebraic genus p for $2 \leq p \leq 6$. In the case of orientable surfaces without boundary, the analogous classification was carried out for surfaces of genus 2 and 3 by Broughton [15] and 4 by Bogopolski [10] and Kimura [51]. In order to find all possible smooth epimorphisms $\Lambda \to G$ for a given group Λ, we used here the computer software MAGMA. For $p = 5$ and 6 we obtained respectively 273 and 216 nonequivalent actions. In [P6, Section 3] we consider also actions of groups of order smaller than 6, but they are too numerous for a complete classification. Instead, for every group of order at most 5 we found all topological types of bordered surfaces of any genus on which this group acts. We also obtained the analogous result for all groups of prime order.

As I already wrote in the introduction, to every compact Klein surface functorially corresponds certain projective real algebraic curve, usually understood as a complex curve defined by a real equation. In view of this correspondence, the results obtained in the papers [P6, P11, P14] can be interpreted as a topological classification of finite group actions on real curves.

3.9. **Branch locus of the moduli space of nonorientable Klein surfaces.** [P10]

Let F be a closed surface of negative Euler characteristic. The moduli space $\mathfrak{M}(F)$ of Riemann or Klein surfaces homeomorphic to F is the orbit space of a properly discontinues action of the mapping class group $\mathcal{M}(F)$ on the Teichmüller space $\mathrm{Teich}(F)$. Since $\mathrm{Teich}(F)$ is a manifold, homeomorphic to a ball in an euclidean space, $\mathfrak{M}(F)$ has the structure of an orbifold. The singular points of $\mathfrak{M}(F)$ correspond to Riemann or Klein surfaces admitting nontrivial automorphisms. The set of all singular points of $\mathfrak{M}(F)$ is called branch locus and is denoted by $\mathcal{B}(F)$.

The study of the branch locus $\mathcal{B}(S_g)$ of the moduli space of Riemann surfaces of genus $g \geq 2$ is a classical problem, whose history goes back to the 1960s. The vast literature devoted to this subject contains a series of papers about connectivity of $\mathcal{B}(S_g)$. The final result is that $\mathcal{B}(S_g)$ is a connected subset of $\mathfrak{M}(S_g)$ if and only if $g \in \{3, 4, 7, 13, 17, 19, 59\}$ [5].

In the paper [P10] we study the branch locus $\mathcal{B}(N_g)$ of closed nonorientable Klein surfaces of genus $3 \leq g \leq 5$. As the main result we proved that $\mathcal{B}(N_g)$ is a connected subset of $\mathfrak{M}(N_g)$ for $g = 4$ and $g = 5$. Connectivity of $\mathcal{B}(N_3)$ was already known. It follows from the fact that all Klein surfaces of genus 3 are hyperelliptic, and hence they admit a nontrivial automorphism.

Similarly as in [5], our proof of connectivity of $\mathcal{B}(N_g)$ is based on a well know stratification of the moduli space, described for example in [14, 34]. With respect to this stratification, $\mathcal{B}(N_g)$ is the union of certain connected subsets of $\mathfrak{M}(N_g)$, corresponding to topological equivalence classes of finite group actions on N_g. Thus the study of connectivity of $\mathcal{B}(N_g)$ is related to the subject described in Section 3.8. This research thread should be continued, in order to find all values of g, for which $\mathcal{B}(N_g)$ is a connected subset of $\mathfrak{M}(N_g)$.

4. Research plans

I close this autopresentation with a description of my research plans in a long time perspective, focusing on the initial steps of each particular thread, where I already have some quite concrete ideas and plans. I will mainly continue my work on the mapping class group of a nonorientable surface, in the directions partially outlined in the description of my scientific achievements. I am also thinking about expanding my research area to natural applications, requiring various skills and tools. Therefore I am counting on a participation of collaborators in the realisation of particular goals, having the preliminary consent of many of them. This will mainly be a collaboration within the existing research group in my home University of Gdańsk (first of all G. Gromadzki and M. Stukow). The project also assumes participation of future Ph.D. students, and its ultimate goal is the foundation of a research group working on a few broad subjects based on the knowledge of mapping class groups of surfaces.

4.1. **The Torelli group of a nonorientable surface.** One of the most important subgroups of the mapping class group of a surface F is the Torelli subgroup $\mathcal{I}(F)$ consisting of the isotopy classes of homeomorphisms inducing the identity on $H_1(F,\mathbb{Z})$. In the case of an orientable surface, the basic results and tools of the study of the Torelli subgroup are due to D. Johnson [47, 48, 49]. Very little is known about the Torreli group of a nonorientable surface. The first significant result about $\mathcal{I}(N)$ was obtained only recently by Hirose and Kobayashi [42], who found certain generating set of $\mathcal{I}(N)$. This set is infinite and one of my goals will be to find a finite generating set of $\mathcal{I}(N)$ and to develop, in the nonorientable setting, an analogue of Johnson's theory of the group $\mathcal{I}(S)$. One of the first specific goals will be the definition of "Johnson's homomorphism" for $\mathcal{I}(N)$, as a step towards the computation of the abalianization of this group in a longer perspective. It seems that the this goal can be approached in the spirit of the paper [H4], using the orientable double cover $S_{g-1} \to N_g$. By Gastesi's theorem [25], which can be obtained as a corollary from my Lemma 4.1 in [H4], $\mathcal{I}(N_g)$ is isomorphic to a subgroup of $\mathcal{I}(S_{g-1})$, and hence we can restrict the Johnson's homomorphism defined on $\mathcal{I}(S_{g-1})$ to a homomorphism $\mathcal{I}(N_g) \to \wedge^3 H_1(S_{g-1},\mathbb{Z})$. The natural questions appear, about the image and generators of the kernel of the above homomorphism. I will also try to define the Johnson's homomorphism for $\mathcal{I}(N)$ without referring to orientable surface. It is worth remarking that Hirose and Sato [41] used the Johnson's homomorphism modulo 2, defined on the level 2 mapping class group $\Gamma_2(N)$ of a nonorientable surface, in their computation of the abelianization of that group, where I also have my own experience and from the papers [H2, H3]. For April 2016 I am planning a one week long visit to the University of Tokyo, at the invitation of professor Nariya Kawazumi and entirely funded from his grant. Professor Kawazumi is an expert on Johnson's homomorphism and I am convinced that a discussion with him will be inspiring. In short, I am counting on a collaboration with experts like S. Hirose and N. Kawazumi in this thread.

4.2. **Torsion generators.** It is known that the mapping class group of a closed surface is generated by elements of finite order. An important property of such elements is that they can be represented by conformal automorphisms of a Riemann surface, with respect to some analytic structure, which allows their analysis by methods of hyperbolic geometry and combinatorial group theory, thanks to the Riemann uniformization theorem. This is a very powerful method, by which C Maclachlan proved simple connectivity of the moduli space of complex algebraic curves [64], and I obtained in [P1] an analogous result for purely imaginary real algebraic curves (these are complex curves having real equations but no $\mathbb{R}$-rational points). In this subject I also have some experience from my Ph.D. thesis. In the paper [P3] I proved that for $g \geq 3$ the group $\mathcal{M}(N_g)$ is generated by 4 involutions, and also is generated by 3 elements, two of which have infinite order. It is an open question, whether this group is generated by 2 elements or by 3 involutions. Another question which I would like to answer is whether $\mathcal{M}(N_g)$ is generated by elements of maximal finite order. If yes, then what is the minimum number of such generators? This question is motivated by a theorem of Korkmaz [55], who proved that the group $\mathcal{M}(S_g)$ is generated by 2 elements of maximal finite order $4g+2$. This thread does not have a high priority for me personally, but I think that it could be a good material for a future Ph.D. student supervised by me.

4.3. **Simplicial complexes associated with nonorientable surfaces.** By the famous theorem of Ivanov [44], the group of automorphisms of the curve complex $\mathcal{C}(S)$ on an orientable surface S is isomorphic to the extended mapping class group $\mathcal{M}^{\diamond}(S)$. This theorem has been generalized to various other simplicial complexes associated to an orientable surface, and recently also to the case of a nonorientable surface [2]. The last result is a motivation for the study of automorphisms and geometric properties of various complexes which can be associated with a nonorientable surface. I have on mind mainly some natural subcomplexes of the curve complex, such as, for example, the complex of separating curves, one-sided curves with nonorientable complement, curves representing a fixed homology class. This again, in my opinion, can be a good material for a future Ph.D. thesis under my supervision.

4.4. **3-dimensional manifolds - finite group actions on handlebodies.** Taking up the subject of 3-dimensional manifolds is for me a natural step, considering the role of mapping class groups of surfaces in this theory (it is enough to mention the Heegaard splittings or the open book decompositions of 3-manifolds). In the first place I will focus my attention on handlebodies, where I will consider also nonorientable handlebodies obtained by attaching twisted handles to a 3-ball. One of the long term goals of this research tread is the development of new methods of construction and classification of finite group actions on handlebodies. This is a classical subject with a vast literature in the orientable case. I am going to try my hand at this subject, including also the case of nonorientable manifolds, using the experience from my work on finite group actions on surfaces (Section 3.8) and continuing the fruitful collaboration with G. Gromadzki. In the realization of this project I am also counting on a collaboration with M. Stukow, who is an expert on the mapping class group of a nonorientable surface like me, and also R. Hidalgo from Chile, who is an expert on Schottky groups.

We will look for an algebraic criterion that could be used to answer two kinds of questions. First, whether an action of a finite group G on a closed surface F, given by a smooth epimorphism (as described in Section 3.8), extends to an action on a handlebody whose boundary is F? Secondly, when two different extensions of the same action are topologically conjugate? Our first task, which we treat as a testing ground, will be a classification, up to topological conjugation, of finite group actions on orientable handlebodies of low genus 2, 3 and 4. The staring point for this task is the classification, up to isomorphism, of finite groups acting on such handlebodies, found in [71], as well as the results concerning the topological classification of finite group actions on closed orientable surfaces of genera 2, 3 and 4 due to Broughton [15], Kimura [51] and Bogopolski [10], who has expressed his interest in participation in this task. The involvement of O. Bogopolski, who is an outstanding expert in the combinatorial group theory, is important for our plans of extension, to the nonorientable setting, of the classical method of constructing actions on handlebodies of the fundamental group of a graph of groups, due to D. McCullough, A. Miller and B. Zimmermann [71].

4.5. **Mapping class group of a nonorientable handlebody.** Another goal of global nature is the study of algebraic properties of the mapping class of a nonorientable handlebody, whose boundary is a nonorientable surface of even genus. Examples of specific tasks include obtaining a finite generating set of this group, and then a finite presentation, by methods similar to those that led to analogous results for orientable handlebodies [80, 85], and by using the experience from my work on presentations of mapping class groups of surfaces. Also in this thread I am counting on a fruitful collaboration with S. Hirose, already initiated during his visit to Gdańsk in June 2015. This subject is essentially completely new, and I think that there is also a lot of space for a future Ph.D. student.

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