Abstract of the thesis

In the thesis we discuss the notion of permutations with supports contained in some ideal. We focus on permutations of multidimensional series and infinite matrices. By support we understand such subset of the domain where the permutation is non-identical and by ideal we understand a family of small subsets of a certain space. In our case all permutations will act on a set of natural numbers ω and all ideals will be collections of subsets of ω. Therefore we consider those permutations which move only a small number of elements.

The thesis consists of five chapters. In the first one we introduce all the necessary definitions and theorems from literature which we are using in our proofs.

In the second chapter we discuss the concept of conditionally convergent series restricted to a small number of indices. This notion was initiated by Wilczyński [W] who generalized a well known theorem of Riemann [Ri] stating that given a conditionally convergent series of reals one can find a rearrangement of the series such that the new sum is equal to a fixed real number. Wilczyński's generalization asserts that one can also require that the permutation moves only a small number of elements – in his case a set of asymptotic density zero. Because such sets form an ideal (usually denoted as $\mathcal{I}_d$) it is natural to ask what other ideals can be put in its place. The problem was solved by Filipów and Szuca in [FS], where they stated that the ideals in question are exactly the ones which cannot be extended to a summable ideal. Such feature was called the (R) property of the ideal.

In our work we undertook a similar subject but this time regarding multidimensional series. The basis for this notion is a theorem proved by Lévy [L] and Steinitz [St] stating that the set of sums of a series obtainable by rearranging the series' elements is a translate of a linear subspace. The exact form of the set depends on the series itself. In the second chapter we focus on $\mathbb{R}^2$ and show that if the set of sums of a series is a line on the plane then using the same ideals as in the case of $\mathbb{R}$ it is possible to rearrange the series so that still any point on the line can be obtained. We also prove that the (R) property is equivalent to the following fact: if the sum of a series is the entire plane then there exists a set from the ideal such that the series restricted to this set can still obtain any point on the plane. This theorem is not entirely analogous to the one-dimensional counterpart as we do not assure the convergence of the non-rearranged series on the set from the ideal. We managed to assert the convergence in the case of at least three Lévy vectors. The case of two Lévy vectors remains an open problem.

In the third chapter we furtherly discuss the (R) property by verifying which of the known ideals have it. In [FS] authors notice that if it turns out that the van der Waerden ideal $\mathcal{W}$ has the (R) property then this fact solves the Erdös-Turán hypothesis stating that $\mathcal{W}$ is contained in the summable ideal $\mathcal{I}_{\left(\frac{1}{n}\right)}$. One of our results is a theorem stating that the van der Waerden ideal is contained in a certain summable ideal, which means it doesn't have the (R) property. Therefore we show that the method above does not solve the Erdös-Turán hypothesis.

In the next part we verify that another known ideal, Hindman ideal $\mathcal{H}$, has the (R) property. In the thesis we present two proofs of this fact, where the second one has been suggested by R. Filipów. Also, a lemma appearing in his argument has been proved by M. Kojman in [Ko], however, we present our own proof.

The third chapter is closed by a series of theorems regarding a cardinal coefficient κ_M defined as the minimal number of summable ideals required to cover a fixed ideal (therefore it is a natural extension of the definition of the (R) property). During the research on this number we encountered a similar approach by Mazur [M]. In his paper he showed that a certain ideal $\mathcal{I}_{ns}$ defined by him (a crucial property of this ideal was being an F_σ ideal not contained in a summable ideal) cannot be covered by λ many summable ideals for any $\lambda < cov(\mathcal{M})$, where $cov(\mathcal{M})$ is the minimal number of meager sets required to cover the real line. Using our coefficient we can say that Mazur showed that $\kappa_M(\mathcal{I}_{ns}) \geqslant cov(\mathcal{M})$. In the third chapter we present basic properties of this number for any ideal.

Then we move to the part of the thesis regarding axial permutations. In 1935 Stefan Banach posed a question in *The Scottish Book*, whether every permutation of ω^2 is a composition of a finite number of axial permutations. A positive answer was given by Nosarzewska in [N], where she stated that a sufficient amount of axial permutations is five. The result was strengthened by Ehrenfeucht and Grzegorek in [EG] and [G], where the number four was put in place of five.

In the fourth chapter we are interested in those axial permutations for which the support on each axis is contained in some ideal. In the entire thesis we assume that all ideals contain $\mathcal{F}in$, the ideal of finite sets, and in particular we study the case where the supports are finite. We prove the strengthened version of the historical results by asserting that every permutation of ω^2 can be represented as a composition of finitely many axial permutations such that each of them has a finite support on each axis. Naturally, a higher requirement regarding the axial permutations increases the minimal number of the sufficient ones. We estimated from above such amount by ninety-six.

In the next part we consider a more general case of axial permutations with supports contained in some ideal. By constructing a counterexample we show that one cannot strengthen the classic result of four axial permutations by additionally requiring that the permutations have supports contained in some ideal on each axis. It shows that increasing the requirements regarding the axial permutations indeed changes the minimal amount of them in the composition.

In the last chapter we consider general maps of ω^2. We ask when it is possible to represent them as a composition of a finite number of axial maps with finite supports on each axis. Unlike the case of axial permutations we show that not every map has such property. In the final part we establish which ones do.

The thesis is based on four papers. Two of them ([K1], [K2]) are already published and the other two ([KN1], [KN2]) are in preparation.

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