Summary of PhD Thesis

I. MOTIVATION AND MAIN GOALS OF PHD THESIS

It is well known that the language of the physics is mathematics, but also something more [1]. Mathematical structures using by physicists with so great success not only describe nature, but also model its internal structure (so called mathematical modeling). Thanks to this incomprehensible up to this days connection we describe not only world around us, i.e. we say how surrounding us things are, but we can also using pure logic considerations to predict some phenomenas, which we can not observe and feel explicitly. It because of mathematical modeling we learn which properties we can neglect constructing some experimental device or how to interpret result of measurements. In this dissertation author together with collaborators raise a series of problems from the new branch of science, namely from quantum information theory. We describe some aspects of group-theoretical description of quantum cloning machines, distillation of entanglement and finally construction of new mathematical tools, which help us with above mentioned mathematical modeling, this time in the micro-scale regime.

Now we describe briefly motivation behind problem of quantum cloning or entanglement distillation but the mathematical point of view. We start from copying of quantum states.

Form the linearity of quantum mechanics we know that perfectly cloning of unknown quantum state is forbidden. It is statement of famous co-cloning theorem given first time by Wooters and Żurek [1] and independently by Dieks [2]. Because preparation of perfect clones is forbidden somehow form the definition we can ask in the opposite way. Namely let us formulate the following question: Is it possible to prepare clones which are close to input state? How close to input state our copy can be - how good copy can we obtain [2]? It turns out that we have positive answers for the first question, reader can find details in [3–7]. Also on the second question we have satisfactory answer, for both

[1]Mentioned philosophical problem is called: "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."

[2]Using word "close" we think about measure, which we called quantum fidelity.

symmetric, universal cloning machines [3] and antisymmetric, universal cloning machines [4] [8–10]. Despite significant progress in this topic in last years, up to our best knowledge nobody gives connection between cloning machines and structure of symmetric group $S(n)$. Explanation of this connection is one of the main goals of this PhD dissertation.

Second quantum informational aspect considered in dissertation is the entanglement distillation. It is well known, that pure entanglement is one of the most important resource in quantum information, we can mention here a few classical results contained in [11–13]. However in the most of the practical cases we have access to mixed entanglement, which is no longer such useful and universal as pure one. To obtain pure entanglement, usually in the form of maximally entanglement pairs we should be able somehow distill them from initial mixture. Procedures which allow us for such filtering we called distillation protocols and they are realized by LOCC [5] operations [14–17]. If two observers share n copies of state consists mixed entanglement, and then they use distillation protocol, then as a result they should get pure entanglement in the form of the state which is close to m ($m < n$) copies of maximally entangled state, and the limit $\lim_{n\to\infty}\frac{m}{n}$ we called efficiency of the protocol. In this dissertation we present some distillation protocol for entanglement distillation for which description we make a use very strong tools form representation theory of symmetric group $S(n)$.

As we can see from above description the main link for all the topics of representation theory of symmetric group $S(n)$ and some of its modifications about which we write later. Because we deal with properties of very large class of objects (groups) it worth to say before main part of this summary a few words about methodology which authors have used here. We start form well known Schur-Weyl duality [18] and then we define some mathematical problem which extends mentioned dualism onto larger class of objects.

Let us consider $n-$partite Hilbert space $\mathcal{H}^{\otimes n}$, wherein we assume that $\mathcal{H} \cong \mathbb{C}^d$ and $d \in \mathbb{N}$ is the dimension of every copy $\mathcal{H}$. It is known that every operator $X : \left(\mathbb{C}^d\right)^{\otimes n} \to \left(\mathbb{C}^d\right)^{\otimes n}$, which commutes with unitary operations of the type

[3] Quantum fidelities of all clones calculated respect to initial state are identical.

[4] Quantum fidelities for all clones calculated respect to initial state do not have to be equal.

[5] Local Operations and Classical Communication

$U^{\otimes n}$, i.e. satisfies relation

$$\left[X, U^{\otimes n}\right] = 0, \tag{1}$$

can be written as some linear combination of permutation operators $\mathrm{V}(\sigma)$:

$$\mathrm{X} = \sum_{\sigma \in S(n)} a(\sigma)\, \mathrm{V}(\sigma), \tag{2}$$

where $a(\sigma)$ for $\sigma \in S(n)$ are some, known coefficients of combination, and operators $\mathrm{V}(\sigma)$ act on the basis vectors $|e_{i_1}\rangle \otimes \cdots \otimes |e_{i_n}\rangle$ of space $\mathcal{H}^{\otimes n}$ as follows

$$\forall \sigma \in S(n) \quad \mathrm{V}(\sigma)|e_{i_1}\rangle \otimes \cdots \otimes |e_{i_n}\rangle = |e_{i_{\sigma^{-1}(1)}}\rangle \otimes \cdots \otimes |e_{i_{\sigma^{-1}(n)}}\rangle. \tag{3}$$

Therefore, to know irreducible components of operator X it is enough to know irreducible components of every operator $\mathrm{V}(sigma)$ separately.

Speaking more precisely, whenever we deal with the problem, that operator X possess property given by the equation (1) we can use representation theory to reduce our problem to study operators $\mathrm{V}(\sigma)$ in the block-diagonal form, i.e.

$$\mathrm{V}(\sigma) = \bigoplus_{\lambda} \mathrm{V}_{\lambda}(\sigma), \tag{4}$$

where direct sum runs over all irreducible representations λ of the symmetric group $S(n)$.

Going further we can use Schur-Weyl dualism [18], which states that duality between irreducible representations of symmetric group and full linear group [6]. Thanks to above-mentioned dualism we write every element of the direct sum (4) as

$$\mathrm{V}_{\lambda}(\sigma) = \mathbb{1}_{\lambda}^{\mathcal{U}} \otimes \mathrm{V}_{\lambda}^{\mathcal{S}}(\sigma), \tag{5}$$

where symbols $\mathcal{U}, \mathcal{S}$ denote these parts of the operator which acts on the unitary and symmetric space respectively. From the equation (5) we see additionally, that every component $\mathrm{V}_{\lambda}(\sigma)$ has tensor structure, wherein non-trivial part acts only on symmetric part. Therefore to learn about irreducible components of the operator X it is enough to know irreducible components of every $\mathrm{V}_{\lambda}(\sigma)$, where multiplicities are given as the dimension of the operator $\mathbb{1}_{\lambda}^{\mathcal{U}}$. Description of the symmetric part, speaking more precisely, way of calculation matrix representations of $\mathrm{V}_{\lambda}^{\mathcal{S}}(\sigma)$ is given by the Young-Yamanouchi construction [19].

[6] Reader notices that in this summary we are interested in subgroup of the $\mathrm{GL}(n, \mathbb{C})$, namely group of the unitary matrices $\mathrm{U}(n, \mathbb{C})$.

Now we can ask, what happen if in the above construction our operator X would be invariant respect to transformation of the type $U^{\otimes(n-k)} \otimes U^{*\otimes k}$. It means when

$$\left[X, U^{\otimes(n-k)} \otimes U^{*\otimes k}\right] = 0, \tag{6}$$

where $*$ denotes complex conjugation. It can be proven, then operator X can be written as a linear combination of permutation operators $\mathrm{V}(\sigma)$, but partially transposed over k last subsystems [7]

$$X = \sum_{\sigma \in S(n)} b(\sigma)\, \mathrm{V}^{\Gamma_k}(\sigma), \qquad \Gamma_k = T_{n-k+1} \cdots T_n, \tag{7}$$

where T_i for $n-k+1 \quad i \quad n$ denote standard transposition on the i^{th} subsystem.

In this dissertation we confine ourselves to the simplest, but non-trivial case, when $k = 1$, i.e. when partial transposition acts on the last n^{th} subsystem. Therefore our task is to find decomposition of the operators $\mathrm{V}^{T_n}(\sigma)$ similarly as in equations (5), (6) and also present construction method of the irreducible matrix representations (Paragraph c).

Having knowledge about decomposition of the operators form formulas (2) and (7) into irreducible components , together with their matrix representations we can use them (full explanation in the next chapter) to group-theoretical and algebraic description of the cloning machines (Paragraphs a,b) or calculation efficiency of some entanglement distillation protocol (Paragraph d).

II. SUMMARY OF RESULTS CONTAINED IN PHD THESIS

Papers consisting for this dissertation, i.e. articles form the positions [A-E] we can divide in three groups. The first group, that is papers [E] and [D] exploit known method from representation theory of finite groups with special emphasis on symmetric (permutation) group $S(n)$ for such problems as distillation of entanglement by projection on permutationally invariant subspaces or group-theoretical description universal, qubit [8] cloning machines. The second group of the papers, form the positions [C], [B] consists construction of the irreducible

[7]It worth to say here, that assumptions about conjugation last k operations U does not decrease generality of the problem. We can always apply proper rotation to obtain demanding form.

[8]Quantum-mechanical system described on two-dimensional Hilbert space.

representations partially transposed permutation operators, generalizing and expanding existing knowledge about permutation operators. Finally the last paper form the series, i.e. position [A] uses previously developed mathematical tools to extend results form [D] for higher-dimensional cases. Now we describe shortly results form every paper starting from the problem of quantum cloning, both qubit and qudit cases (Paragraphs a,b), then we focus on the construction of some special mathematical tools (Paragraph c) and we end on the mathematical problem connected with distillation of entanglement (Paragraph d).

a. **Group-theoretical approach to universal quantum cloning machines** In the paper [D] we present group-theoretical approach to the problem when states which we want to clone are maximally entangled qubit states, for example one of the Bell sate of the form [9]:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \tag{8}$$

The main goal of the paper [D] is description in the analytical way allowed range of fidelities for the $1 \to N$ universal, qubit cloning machines [10], which follows from the laws of quantum mechanics by the application representation theory of symmetric group $S(n)$. Speaking more precisely, we want to find analytical constraints for the following quantities:

$$F_{1i} \equiv F\left(\Phi^+, \operatorname{Tr}_{\overline{1i}}\rho_{1\ldots n}\right) = \operatorname{Tr}\left[\sqrt{\sqrt{\Phi^+}\operatorname{Tr}_{\overline{1i}}\rho_{1\ldots n}\sqrt{\Phi^+}}\right], \tag{9}$$

where $\Phi^+ = |\psi^+\rangle\langle\psi^+|$, quantity $\rho_{1\ldots n}$ is the joint state after application of the cloning machine and $\operatorname{Tr}_{\overline{1i}}\rho_{1\ldots n}$ reduced state of i^{th} copy [11]. For the analysis of the problem we use knowledge which are described shortly in the first chapter of this summary. It turns out that in the two-dimensional case (qubits) maximally entangled singlet states (see equation (8)) are $U \otimes U$ invariant. This property allows us to write joint state after application of the cloning machine as

$$\rho_{1\ldots n} = \bigoplus_{\lambda} \mathbb{1}^{\mathcal{U}}_{r(\lambda)} \otimes \widetilde{\rho}^{\lambda}, \tag{10}$$

which is also unitary invariant respect to unitary transformations $U^{\otimes n}$. In the equation (10) by $r(\lambda)$ we denote the dimension of the irreducible representation

[9] Of course we can use also other Bell states.

[10] In this dissertation we use the following convention: Number of the copies we denote by capital N, but $n = N + 1$, where n is the degree of the symmetric group $S(n)$.

[11] By the $\operatorname{Tr}_{\overline{1i}}\rho_{1\ldots n}$ we denote partial trace over all subsystems, except the first one and the i^{th} one.

corresponding with partition λ, while $\widetilde{\rho}^{\lambda}$ is irreducible representation of the density operator $\rho_{1\ldots n}$ acting on symmetry part. Thanks to this property, we show that calculating quantum fidelity F_{1i} between qubit input sate ρ and i^{th} copy can be reduced to calculating fidelity on the permutationally invariant subspace, which follow from the decomposition of the permutation operators $\mathrm{V}(\sigma)$ into irreducible components (Lemma 1):

$$F_{1i} = \sum_{\lambda} F_{1i}^{\lambda}, \quad \text{where} \quad F_{1i}^{\lambda} = \frac{1}{2} - \frac{1}{2}\operatorname{Tr}\left(\rho^{\lambda}\,\mathrm{V}_{\lambda}^{\mathcal{S}}(1i)\right), \tag{11}$$

where $\mathrm{V}_{\lambda}^{\mathcal{S}}(1i)$ denotes irreducible representation of the permutation operator $\mathrm{V}(\sigma)$, when $\sigma = (1i)$. Next important step on the way to solution is the observation, that to obtain whole allowed range of fidelities it is necessary to take convex hull from the set of all calculated fidelities for all possible copies $1 < i \ \ n$ and representations λ (Theorem 1):

$$\mathcal{F} = \operatorname{conv}\left(\bigcup_{\lambda}\left\{\left(F_{12}^{\lambda}, \ldots, F_{1n}^{\lambda}\right) \ : |\psi\rangle \in \mathbb{C}^{d_{\lambda}} \ \right\}\right). \tag{12}$$

The main part of the paper [D] ends with the proof of Lemma 3, which states that to generate convex hull from the Theorem 1 is necessary and sufficient to take only the real states. In other words we have shown some kind of majorization of the complex states by the real ones in the case of such type of cloning machines. In the paper [D] we put additionally graphical representations for allowed rage of fidelities in the case of $1 \to 3$ cloning machines, for every irreducible representation separately (Figure 2) and after generation of the convex hull (Figure 3). We show also how we can apply described method to reconstruction quantum states with certain constraints of fidelities, in the case allowed by theory (Chapter VF).

b. **Algebraic description of cloning machines** The paper [A] is the generalization of the results from [D] for the case when as the input state for cloning we take maximally entangled qudit [12] state

$$|\psi^{+}\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle, \tag{13}$$

where d is the dimension of Hilbert space on which our state is described.

It turns out that in the case $d > 2$ we can not apply directly known tools from representation theory of symmetric group $S(n)$ as in the qubit case. This situation follows from the fact, that states from the equation (13), for $d > 2$ are no

[12] Quantum-mechanical system described on $d-$dimensional Hilbert space.

longer $U \otimes U$ invariant, but there are $U^* \otimes U$ invariant [13]. Different symmetry in the higher-dimensional case causes, that the joint state $\rho_{1\ldots n}$ after application of the cloning machine is invariant respect to transformations $U^* \otimes U^{\otimes(n-1)}$. If we still want to have representation approach to the problem the necessary is knowledge about irreducible components of the algebra [14] of partially transposed permutation operators $\mathrm{V}^{T_n}(\sigma)$. It is clear now that the crucial role here play mathematical tools which are described in the Paragraph c) of this summary. Authors of the paper [D] show, that fidelities between input state and an arbitrary clone can be written as irreducible representations of partially transposed permutation operators $\mathrm{V}_{\alpha}^{T_n}(k-1\ n)$, where $1 < k \ \ n$ obtained in [C], [B] and density operator ρ^{α} acting on particular irreducible subspace labelled by partition α. Namely we have (Lemma 1):

$$F_{1k}^{\mathcal{M}} = \sum_{\alpha} F_{1k}^{\alpha}, \quad \text{where} \quad F_{1k}^{\alpha} = \frac{1}{d} \operatorname{Tr}\left[\rho^{\alpha}\, \mathrm{V}_{\alpha}^{T_n}(k-1\ n)\right], \tag{14}$$

where d is the dimension of the Hilbert space. We have to stress here that we introduce index α instead of λ as in the qubit case to distinguish appropriate representations. From the papers [B] and [C] we know that algebra $\mathcal{A}_n^{t_n}(d)$ splits into sum of two spaces $\mathcal{M}$ and $\mathcal{A}_{n-1}(d)$, wherein formulas (14) describe fidelities calculated for the elements from the ideal $\mathcal{M}$. In the case of the ideal $\mathcal{A}_{n-1}(d)$ formula for fidelities reduces to the simply equation:

$$F_{1k}^{\mathcal{N}} = \frac{1}{d}. \tag{15}$$

Further, similarly as for qubit case authors show, that to obtain allowed region of fidelities for $1 \to N$ universal qudit cloning machine is necessary to take convex hull from the set of all possible copies $1 < i \ \ n$ and representations α (Theorem 3). Moreover they argue that still majorization of the complex states by the real ones in the sense of fidelity is valid (Lemma 4).

c. **Irreducible representations of partially transposed permutation operators** For this part of dissertation consist papers [C] and [D]. They show equivalent approach for the same problem, namely to problem of finding irre-

[13]By $*$ we denote complex conjugation.

[14]Reader notices that the set of permutation operators creates group, in particular every element have inverse. However the set of partially transposed permutation operators creates an algebra, so there are some element for which inverse does not exist.

ducible representations of the algebra $A_n^{T_n}(d)$ partially transposed permutation operators $V^{T_n}(\sigma)$, which were described in the first part of this summary.

Although these two papers treat on the same issue, but they give different insight into structure of the problem, and moreover article [B] is important extension results from [C] for low-dimensional cases and gives us connection with the structure of some induced group. Let us say now a few more words about problems connected with these two approaches. We start from the paper [C].

In this article we start from the level of Hilbert space and construction of vector bases spanning every irreducible subspace (Definition 4). This allows us to construct non-orthogonal operator basis in Hilbert-Schmidt product (Definition 5, Lemma 8). At this level the crucial observation is the fact, that there is a some set of mappings $\mathcal{F}_{ab}^t$, which connects elements of the algebra $\mathbb{C}[S(n-2)]$ with the elements of $\mathcal{M}$:

$$\mathbb{C}[S(n-2)] \ni V(\sigma) \xmapsto{\mathcal{F}_{ba}^t} V'(\sigma_{ab}) \in \mathcal{M}. \tag{16}$$

Form the linearity of mappings $\mathcal{F}_{ab}^t$ we know, that following is true:

$$\mathbb{C}[S(n-2)] \ni E_{ij}^{\alpha} \xmapsto{\mathcal{F}_{ba}^t} v_{ij}^{ab}(\alpha) \in \mathcal{M}, \tag{17}$$

where operators E_{ij}^{α} are well defined Wigner operators for the symmetric group $S(n-2)$. Above formulas imply directly Theorem 9, which is one of the crucial results in described paper. This theorem states how to represent partially transposed permutation operators by non-orthogonal operator basis and vice versa, and also allows us to calculate action of $V^{T_n}(\sigma)$ on mentioned operator basis.

Unfortunately as we have mentioned above, our basis is non-orthogonal, so we can not to calculate matrix representations of $V^{T_n}(\sigma)$ in the sense how we used to in physics. To solve this problem we use matrix $Q(\alpha)$ (Definition 11), which is Gramm matrix of the basis vectors from the Definition 4. This matrix is block-diagonal and possess a lot of interesting properties, which determine our further approach. Namely blocks of the matrix $Q(\alpha)$ are the matrix representations appropriate transpositions (Definition 11, Remark 12) and what is the most important, because this is the Gram matrix it can happen that for certain relations between dimension d of the Hilbert space and number of subsystems n our set of vectors in linearly dependent. This implies that matrix $Q(\alpha)$ is non-invertible for some cases. Detailed analysis of this problem is given in the

Theorem 13, which states that whenever $d > n-2$ Gramm matrix is always invertible. It turns out that whenever we are in the regime $d > n-2$ we can redefine by matrix $Q(\alpha)$ our non-orthogonal operator basis to obtain orthogonal one in the Hilbert-Schmidt product (Definition 14). In the next step we define left action of the new, orthogonal basis operators on $\mathrm{V}^{T_n}(\sigma)$ (Proposition 16) and present the procedure of calculating desired matrix elements of irreducible representations (Lemma 18).

Paper [C] says also a few words about case when $d \;\; n-2$ (Chapter IV.B), but it does not give us satisfactory answer in the language of vector basis of Hilbert space. Authors show, that to construction of irreducible representations it is necessary and sufficient to choose from the set of linearly dependent vectors some subset which is linearly independent and next make an argumentation as in the case when $d > n-2$. It causes obvious reduction of the dimension our new operator basis (Example 19). Unfortunately this procedure does not present any effective method how to choose properly linearly independent set of vectors and also how to connect this with the global properties of the algebra $A_n^{T_n}(d)$.

Different approach to the problem of finding irreducible representations of partially transposed permutation operators $\mathrm{V}^{T_n}(\sigma)$ which fills the gap of small dimensions d with respect to the number of subsystems n is presented in [B]. In this paper authors treat algebra $A_n^{T_n}(d)$ in the abstract way and use some advanced algebraic techniques. The most important property in this approaches is to notice, that algebra $A_n^{T_n}(d)$ consists sub-algebra $A_{n-1}(d)$ generated by operators representing sub-group $S(n-1) \subset S(n)$, which are not deformed by partial transposition T_n. This observation allows us to split algebra $A_n^{T_n}(d)$ into two sum of two subspaces:

$$A_n^{t_n}(d) = \mathcal{M} + A_{n-1}(d), \tag{18}$$

where subspace $\mathcal{M}$ is the ideal generated by the operators $\mathrm{V}^{T_n}(\sigma)$, when permutation σ acts on n in non-trivial way [15]. Additionally elements generating ideal $\mathcal{M}$ are non-invertible, which shows that adding even only one partial transposition changes structure of the problem a lot.

Another very important result contained in paper [B] is the connection of the

[15]i.e. when $\sigma(n) \neq n$

structure of the algebra $A_n^{T_n}(d)$ with the structure of the induced representation $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$ of the group $S(n-1)$ induced by irreducible representations φ^{α} of $S(n-2)$ (Chapter IV). More precisely authors show, that all eigenvalues of the matrix $Q(\alpha)$ are labelled exactly by irreducible components of $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$ and their multiplicities are equal to dimension of such representation (Theorem 31). In the next, fifth chapter construction of irreducible representation of the operators $V^{T_n}(\sigma)$ is presented. The starting point is Definition 39, where we define the set of generators $\{u(\alpha)\}$ of the ideal $\mathcal{M}$ together with the composition law and the form of left action on operators $V^{T_n}(\sigma)$ (Proposition 43). Further, similarly as in [C] authors move to the new set of generators, which posses required property of orthogonality (Definition 48) and also present formulas for the matrix elements of irreducible representations of the $V^{T_n}(\sigma)$ operators (Proposition 52) using facts for the generators $u(\alpha)$. The crucial point of this paper is the extension results form article [C] for the case when $\det Q(\alpha) = 0$, i.e. when matrix $Q(\alpha)$ does not have inverse. It means, that some generators $u(\alpha)$ are linearly dependent. Authors present general construction (Theorem 77), which follows to the reduced basis and gives us constructive approach to both, linearly dependent case as well to generic case, when rank of the matrix $Q(\alpha)$ is maximal. This result is based on diagonalization of the matrix $Q(\alpha)$ (Theorem 59) and next, use it for the construction new, consists non-generic case set of generators. It turns out that whenever $\det Q(\alpha) = 0$, then only for one (up to multiplicity) irreducible representation of the group $S(n-2)$ appearing in $\operatorname{ind}_{S(n-2)}^{S(n-1)}(\varphi^{\alpha})$ corresponds zero eigenvalue of $Q(\alpha)$. Then of course generators which correspond to this zero eigenvalue are zero operators and the rest of the span above mentioned, reduced operator basis. At the end it is worth to say that authors present also relative simply algorithm of finding all eigenvalues of the matrix $Q(\alpha)$ and "rejection" of the linearly dependent generators (Appendix A) based on Frobenius theorem [18]. Thanks to this we obtain the full description of the algebra $A_n^{T_n}(d)$ together with the construction of irreducible components and discussion of all its properties with explicit connection to induced group.

d. **Entanglement distillation by projection on permutationally invariant subspaces** In the paper [E] authors consider distillation problem of quantum entanglement form the two-qubit state, which is mixture if two pure, entangled states and one product state orthogonal to them:

$$\rho_{AB} = x\rho'_{AB} + (1-x)|01\rangle\langle 01|_{AB}, \quad x \in [0,1], \tag{19}$$

where ρ'_{AB} is in general non-equal mixture of two states of the form:

$$|\Phi^{\pm}(p)\rangle_{AB} = \sqrt{p}|00\rangle_{AB} \pm \sqrt{1-p}|11\rangle_{AB}, \quad p \in [0,1]. \tag{20}$$

In the considerations we assume, that n copies of the state ρ_{AB} is shared between two peoples, called Alice (lower index A) and Bob (lower index B). After application of proper protocol (Chapter II) our task is to fond analytical expression for the eigenvalues of density operator, which form follows from the construction of the protocol. Mentioned state after measurement has a form:

$$\rho^{(n)}_{lAB} = \frac{P_{lA} \otimes P_{lB}\rho^{\otimes n}_{AB} P_{lA} \otimes P_{lB}}{\mathrm{Tr}(P_{lA} \otimes P_{lB}\rho^{\otimes n}_{AB} P_{lA} \otimes P_{lB})}, \tag{21}$$

where n denotes number of copies of the state ρ_{AB} shared between Alice and Bob. Projectors P_{lA}, P_{lB} project onto subspaces $\mathcal{H}^{(n)}_l$ of space $(\mathbb{C}^2)^{\otimes n}$ spanned by vectors of the standard basis with Hamming weight equal to l, i.e. vectors posses l ones and $n-l$ zeros. Canonical example of such vector is $|\underbrace{0\ldots0}_{n-l}\underbrace{1\ldots1}_{l}\rangle$. The knowledge spectrum of the state operator $\rho^{(n)}_{lAB}$ allows us to calculate efficiency of distillation protocol R_i, which is proportional to coherent information $I_c\left(\rho^{(n)}_{lAB}\right)$. Further we show that density matrix $\rho^{(n)}_{lAB}$ can be represented as a linear combination of the operators $A^{(l)}_k$ acting on the subspaces $\mathcal{H}^{(n)}_l$ spanned by the vectors with definite number of ones (Lemma 1). Therefore knowledge about spectrum of all operators $A^{(l)}_l$ guarantees knowledge about spectrum of density operator $\rho^{(n)}_{lAB}$. Second important observation on the way to final result is fact, that subspaces $\mathcal{H}^{(n)}_l$ are invariant respect to acting of the permutation group $S(n)$. This fact allows us to write every subspace $\mathcal{H}^{(n)}_l$ as a direct sum of irreducible subspaces labelled by Young diagrams maximally with two rows (Lemma 4). Together with some complicated combinatorics and group-theoretical statements contained in Proposition 3, Lemma 5 and Lemma 6 we can prove formula (41) from the Theorem 1, which represents analytical recipe for eigenvalues of the operators $A^{(l)}_k$ and immediately gives us also spectrum of operator $\rho^{(n)}_{lAB}$.

III. FURTHER PERSPECTIVES

At the end of this dissertation summary it is worth to say a few words about further possibilities of investigations. More of them are somehow extensions of ideas from previous sections. The first, and the most direction of future investigation is looking for irreducible representations of partially transposed permutation operators $V^{\Gamma_k}(\sigma)$ in the case of larger number of single transpositions as it was in papers [C] and [D]. We assume of course, that partial transposition has a form $\Gamma_k = T_{n-k+1} \circ \cdots \circ T_n$, where every T_i for $n - k + 1 \quad i \quad n$ denotes standard transposition on i^{th} subsystem. Possibly result will extend knowledge not only about representation theory onto new class of objects, but also gives us interesting applications in quantum information theory. Let us mention here about two possibilities.

Let us start form the possibility of description $N \to M$ universal quantum cloning machines, when we have N input states and M clones (of course we have to assume that $M < N$ and $M + N = n$). Main goal of this problem would be similarly as in [A,D] analytical description of allowed region of fidelities which follow from the laws of quantum mechanics by using again representation theory.

The second one, much more complicated task would be application to additivity problem of quantum channels. Knowledge about irreducible representations of the operators $V^{\Gamma_k}(\sigma)$ allows us to obtain analytical expressions for minimal output Rényi entropy for two copies of the channel. Every such channel would be defined on permutationally invariant subspaces suggested by the Schur-Weyl duality. Thanks to this we should be able to answer for which combinations of parameters as local dimension of Hilbert space d, number of subsystems n and for which subspaces we obtain desired additivity violation.

data i podpis doktoranta

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